Box Models: Stocks, Flows, and Residence Times

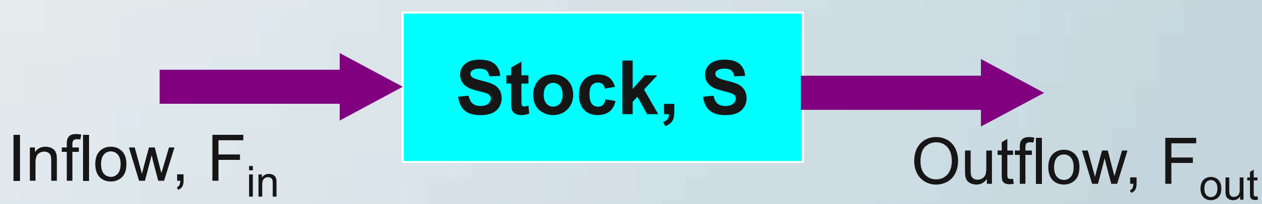
IPOL 8512

Reservoirs

- Natural systems can be characterized by the transport or transformation of matter (e.g. water, gases, nutrients, toxics), energy, and organisms in and out of a reservoir.
- Reservoirs can be physical (e.g. a human body, the atmosphere, ocean mixed layer), chemical (different chemical species), or biological (e.g. populations, live biomass, dead organic matter)
- Transport/transformation can involve bulk movement of matter, diffusion, convection, conduction, radiation, chemical or nuclear reactions, phase changes, births and deaths, etc.

Box Models

- We make a simple model of a system by representing the reservoirs with a “box” and the transport/transformation with arrows.
- We usually assume the box is well-mixed, and we usually are not concerned with internal details.



- *Stock* = the amount of stuff (matter, energy, electric charge, chemical species, organisms, pollutants, etc.) in the reservoir
- *Flows* = the amount of stuff flowing into and out of the reservoir as a function of time

Why Use Box Models?

To understand or predict:

- Concentrations of pollutants in various environmental reservoirs as a function of time: e.g. water pollution, outdoor/ indoor air pollution
- Concentrations of toxic substance in organs after inhalation or ingestion; setting standards for toxic exposure or intake
- Population dynamics, predator-prey and food-chain models, fisheries, wildlife management
- Biogeochemical cycles and climate dynamics (nutrients, energy, air, trace gases, water)

How Box Models Work

Basic rule of box models:

change in stock over time = inflow – outflow

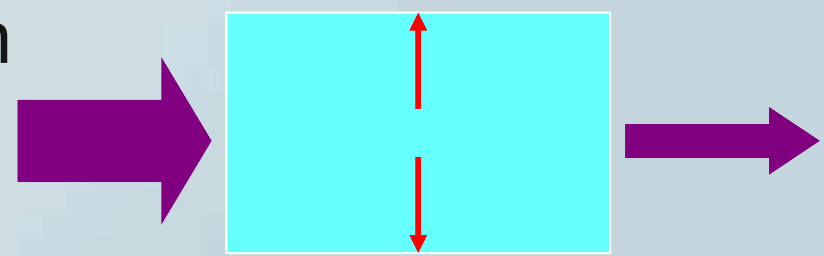
$\Delta S/\Delta t = F_{in} - F_{out}$

Situation 1: Equilibrium

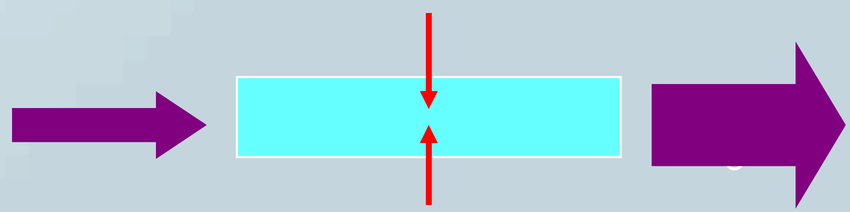
$F_{in} = F_{out}$ $\Delta S/\Delta t = 0$

Situation 2: Non-equilibrium

$F_{in} > F_{out}$ $\Delta S/\Delta t > 0$



$F_{in} < F_{out}$ $\Delta S/\Delta t < 0$



Equilibrium: The Balance

- In many problems in environmental science, it is reasonable to start with the assumption that a particular stock is in equilibrium, meaning that *the stock does not change over time*. In this case, we have

$$F_{IN} - F_{OUT} = 0 \rightarrow F_{IN} = F_{OUT} = F$$

- Three types of equilibria
  - static equilibrium: $F_{IN} = 0$, $F_{OUT} = 0$ (nothing is happening)
  - steady state: $F_{IN} = F_{OUT} =$ constant (nothing is changing)
  - dynamic equilibrium: $F_{IN}(t) = F_{OUT}(t)$ (the changes balance)

We can also make a further distinction between stable equilibria (which tend to persist) and unstable equilibria (which are transitory).

Equilibrium: Three Questions to Ask Yourself

(1) If this stock is in equilibrium, what is implied about the flows involved?

e.g., if the stock of water in the atmosphere is in equilibrium, $F_{in} = F_{out}$, then precipitation must equal evaporation.

(2) If the flows are not in balance, i.e., $F_{IN} \neq F_{OUT}$, how will this manifest itself in the stock?

e.g., what happens to a lake if the water flows in and out of the lake are imbalanced over time, e.g. $F_{OUT} > F_{IN}$?

(3) Over what time period is this stock in equilibrium, and over what time period is it not in equilibrium?

e.g. the stock of water in this lake is currently in balance year to year, but it changes from day to day or month to month or season to season, and it also changes over decades and centuries.

Stock, Flow, and Residence Time

In equilibrium, inflows balance outflows, so stock S doesn't change

$$\Delta S/\Delta t = F_{IN} - F_{OUT} = 0 \qquad S = \text{constant}$$

The flow is the same whether it is inflow or outflow:

$$F_{IN} = F_{OUT} = F$$

The equilibrium stock S can be calculated by the flow F times the residence time T:

equilibrium stock = flow • residence time

$$S = F \cdot T$$

The residence time is the average length of time that the substance spends in the box.

Example

- Consider the stock of students in a four-year college that in which 1,000 freshman per year enroll and 1,000 seniors per year graduate (nobody flunks out or withdraws!) Thus

  $F_{IN} = F_{OUT} = F = 1{,}000$ students/yr

- So the stock S (student population at the college) is constant. How big is it? Remember we know

  equilibrium stock = flow x residence time (S = FT)

- If no one flunks or withdraws, the residence time is 4 years. So we can calculate the stock to be

  $S = FT = 1{,}000 \text{ students/yr} \times 4 \text{ yr} = 4{,}000 \text{ students}$

$$? \text{ students} = 4 \cancel{\text{yr}} \left(\frac{1000 \text{ students}}{1 \cancel{\text{yr}}} \right) = \mathbf{4000 \text{ students}}$$

- If one knows any two of the three quantities -- S, F, T -- then one can always calculate the third: $S = F \bullet T$, $F = S/T$, $T = S/F$

Stocks and Flows of Fresh Water

Stocks

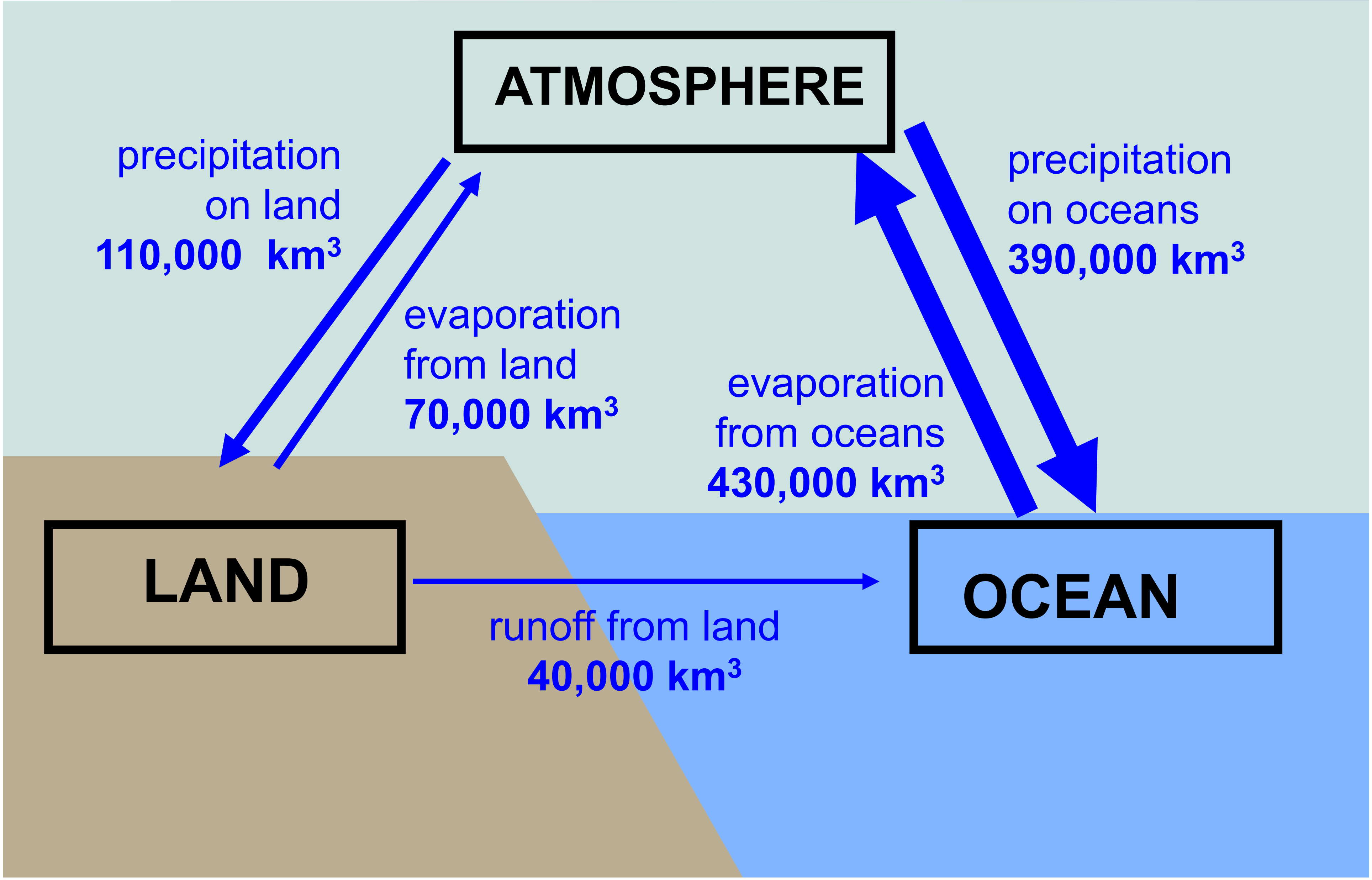
total freshwater	=	40,000,000 km^3
ice	=	30,000,000 km^3
groundwater	=	10,000,000 km^3
lakes	=	100,000 km^3
atmospheric water	=	10,000 km^3
river channels	=	1,000 km^3

These are order of magnitude; see Cow VI.2 for more precise values

Flows:

annual rainfall	=	500,000 km^3/y
annual runoff	=	40,000 km^3/y

Hydrologic cycle box model: flows



Residence Time:
Ex. Water in the Atmosphere

- What is the mean residence time of water in the atmosphere?

$$T = \frac{S}{F} = \frac{10 \times 10^3\ \text{km}^3}{500 \times 10^3\ \text{km}^3/\text{yr}} = (1/50)\ \text{yr} \sim 1\ \text{week}$$

$$?\ \text{week} = 10 \times 10^3\ \cancel{\text{km}^3} \left(\frac{1\ \cancel{\text{yr}}}{500 \times 10^3\ \cancel{\text{km}^3}} \right) \left(\frac{52\ \text{weeks}}{1\ \cancel{\text{yr}}} \right) \approx \mathbf{1\ week}$$

(using better numbers (see COW), T ~ 9 days)

- Significance?
  - Wash-out period for the atmosphere
  - Deposition of water-soluble pollutants

Residence Time: Water in the Ocean

- Mean residence time for oceans?
- Stock / Flow = T, which gives

$$T = \frac{S}{F} = \frac{1400 \times 10^6 \text{ km}^3}{430 \times 10^3 \text{ km}^3/\text{yr}} \sim 3000 \text{ years}$$

$$? \text{ day} = 1400 \times 10^6 \cancel{\text{km}^3} \left(\frac{1 \text{ yr}}{430 \times 10^3 \cancel{\text{km}^3}} \right) = 3256 \approx \mathbf{3000\ yr}$$

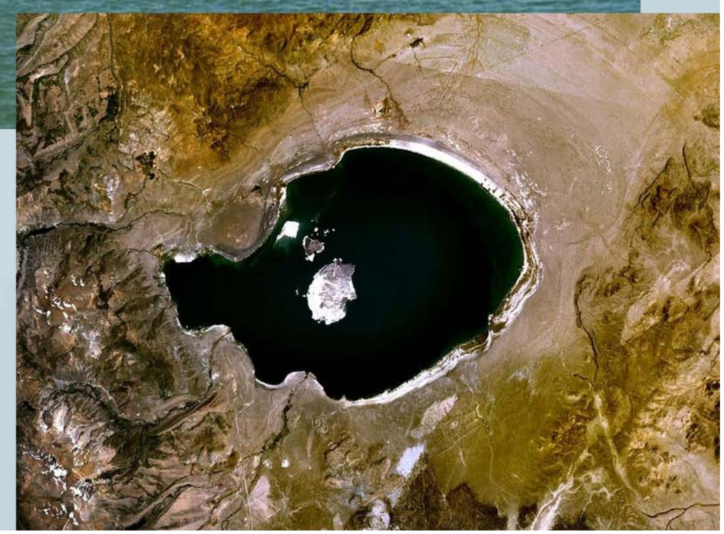
- Mean residence time for mixed layer?

$$T = \frac{S}{F} = \frac{30 \times 10^6 \text{ km}^3}{430 \times 10^3 \text{ km}^3/\text{yr}} \sim 70 \text{ years}$$

$$? \text{ day} = 30 \times 10^6 \cancel{\text{km}^3} \left(\frac{1 \text{ yr}}{430 \times 10^3 \cancel{\text{km}^3}} \right) \approx \mathbf{70\ yr}$$

(Note than this is via exchange with atmosphere and land only; there is another exchange between mixed layer and deep ocean, with an analogous calculation)

Mono Lake and the Use of Box Models in Environmental Science



Mono Lake

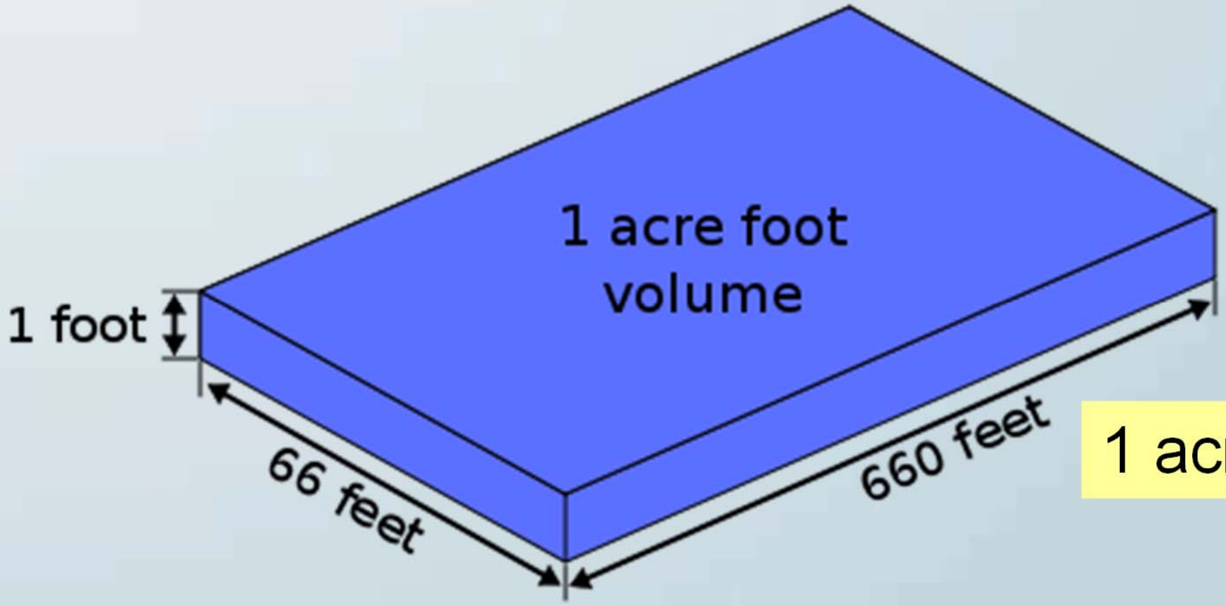
- Mono Lake is a high-altitude basin lake (i.e. no surface outflow, high salinity) on the east side of the Sierra Nevada.
- Dissolved salts in the runoff remain in the lake and raise the water's pH levels and salt concentration.
- Its brine shrimp and islands provide important habitat for two million migratory birds per year.

Mono Lake

- Starting in the 1940s, water diversions to Los Angeles led to a 50 foot decrease in water level.
- By 1982 the lake was reduced to 37,688 acres (15,252 ha) having lost 31 percent of its 1941 surface area. As a result alkaline sands and once-submerged tufa towers became exposed and Negit Island became land-bridged, exposing the nests of gulls to predators (chiefly coyotes) and forcing the breeding colony to abandon the site.
- Public outcry finally forced LA to restore the flows ca. 1990.
- Before 1941, average salinity was approximately 50 grams per liter (g/l) (compared to a value of 31.5 g/l for the world's oceans). In January 1982, when the lake reached its lowest level, the salinity had nearly doubled to 99 g/l. In 2002, it was measured at 78 g/l and is expected to stabilize at an average 69 g/l as the lake replenishes over the next 20 years.

Mono Lake

- Elevation = 6417 ft
- Watershed area = 432,000 acres (432 kac)
  - 2.47 acre = 1 hectare (ha) = 10^4 m^2
- Lake area in 1940 = 55,000 acres (55 kac)
- Lake volume in 1940 = 4300 kac-ft
- Lake volume in 1990 = 2200 kac-ft

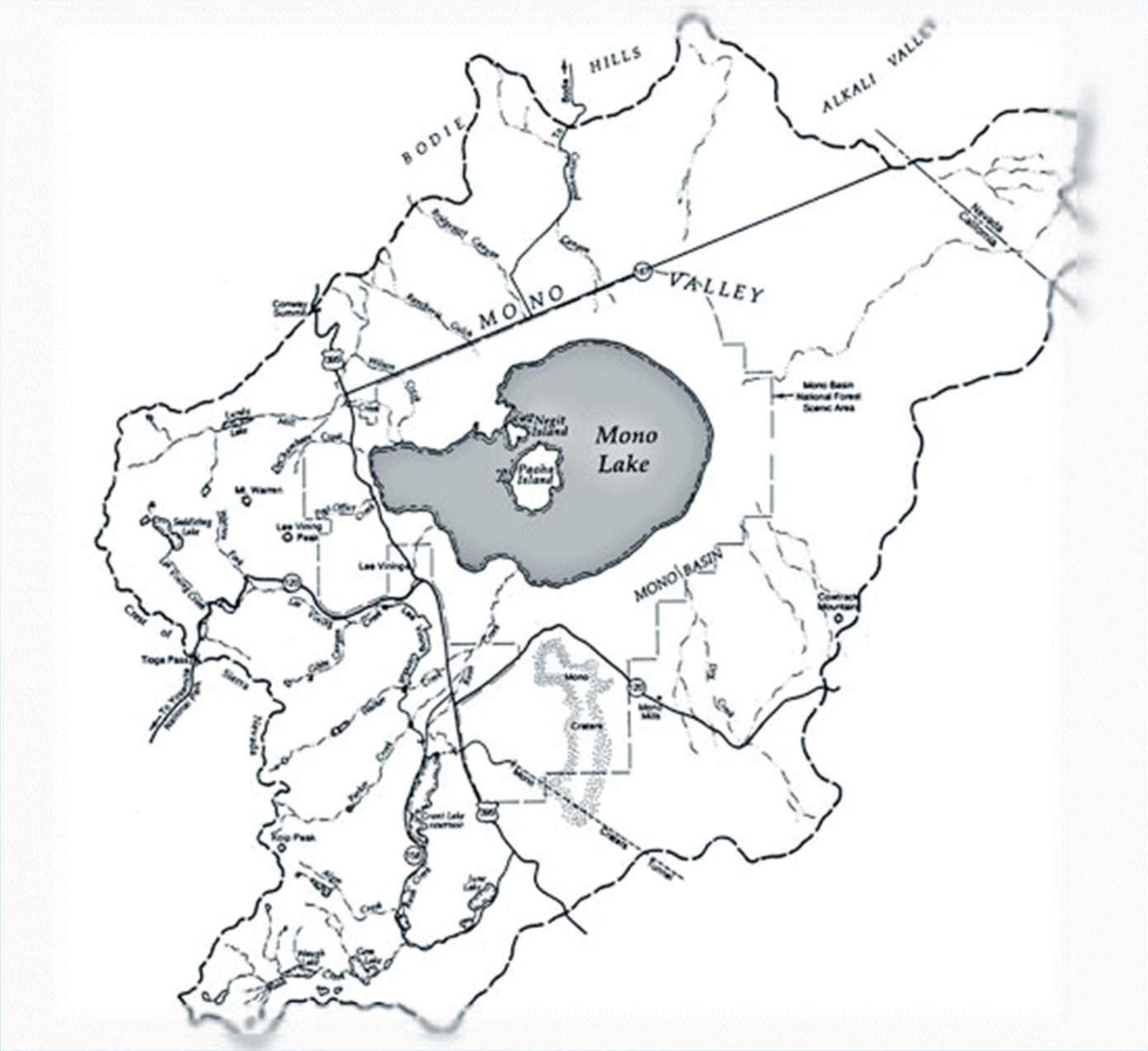


1 acre-foot = 1233 m^3

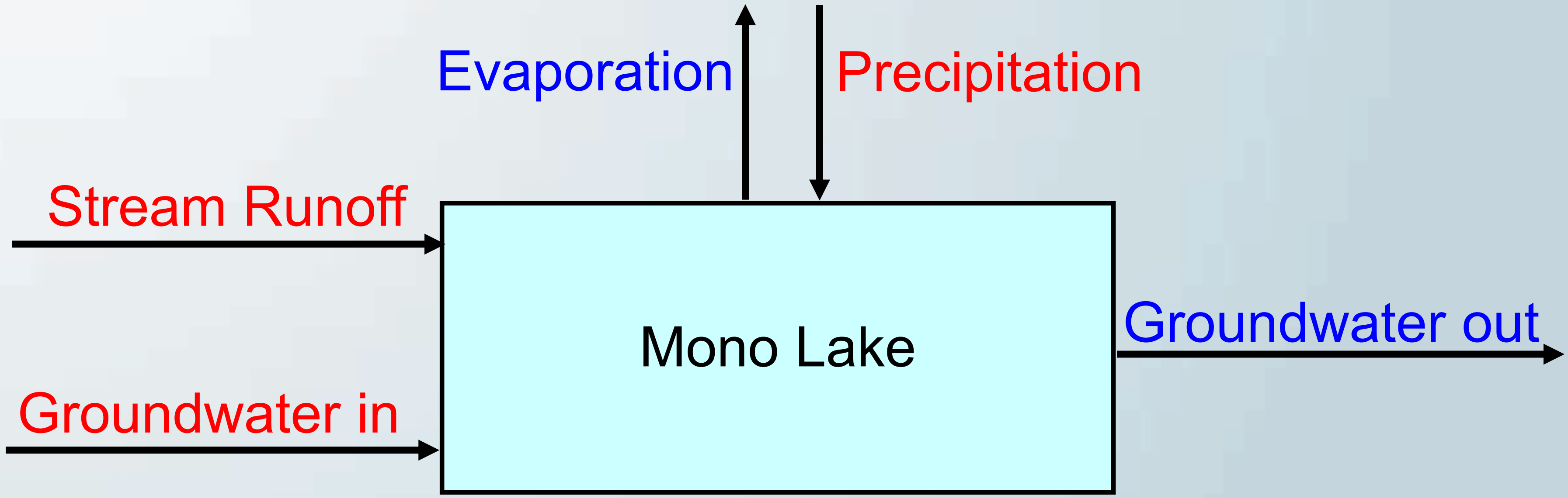
Mono Lake

- Box models were crucial in helping Mono Lake advocates make their case.
- Box model questions:
  - What were the steady-state flows in 1940, before the diversions began?
  - What was the change in lake area, 1940 to 1990?
  - What would happen if the diversions continued?

Mono Lake Watershed



Box Model of Mono Lake Hydrology



F_{in} = runoff inflow + groundwater inflow + precipitation

F_{out} = groundwater outflow + evaporation

Mono Lake Flows

Measured values for Mono Lake flows before 1940, with lake in quasi-equilibrium ($F_{in} \approx F_{out}$):

F_{in} = stream inflow + groundwater inflow + precipitation

- Groundwater inflow = 40 kac-ft/y
- Stream inflow = we will estimate
- Precipitation = 8 in/y (we need to convert to kac-ft/y)

F_{out} = groundwater outflow + evaporation

- Groundwater outflow = 25 kac-ft/y
- Evaporation = ?

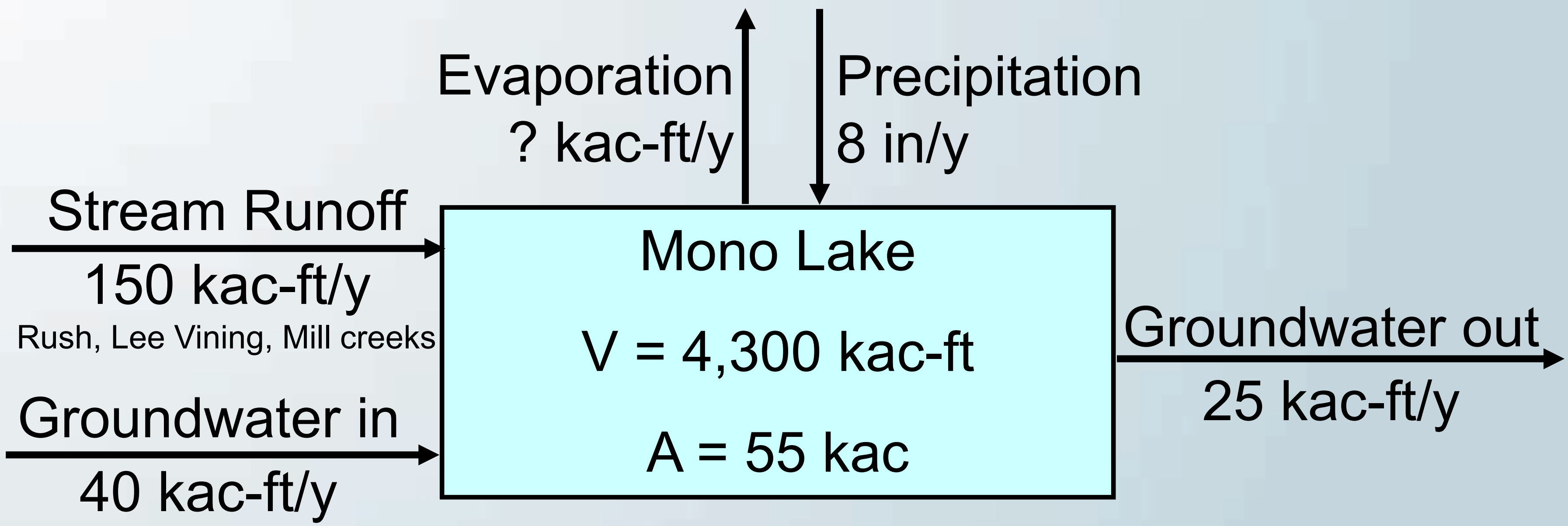
Task: estimate stream inflow from data for Mono Lake watershed, then use box model to estimate evaporation.

Ex. Estimating Runoff from Mono Basin

- For any watershed (drainage basin): $F_{in} = F_{out}$
  Precip = Evap + Runoff + Groundwater Recharge
  $P = E + R + G \rightarrow R = P - E - G$
  Measured average annual values for Mono Basin:
- P = 21 in/y
- E = 8.4 in/y
- G = 7.4 in/y
- R = 21 - 8.4 - 7.4 = 5.2 in/y = 0.43 ft/y
- Watershed area (minus lake) = 432 kac – 55 kac = 377 kac
  ∴ Runoff = (377 kac)(0.43 ft/y) = 160 kac-ft/y

Measured flow into lake ca. 1940 ~150 kac-ft/y

Ex. Using Box Model to Calculate Evaporation from Mono Lake



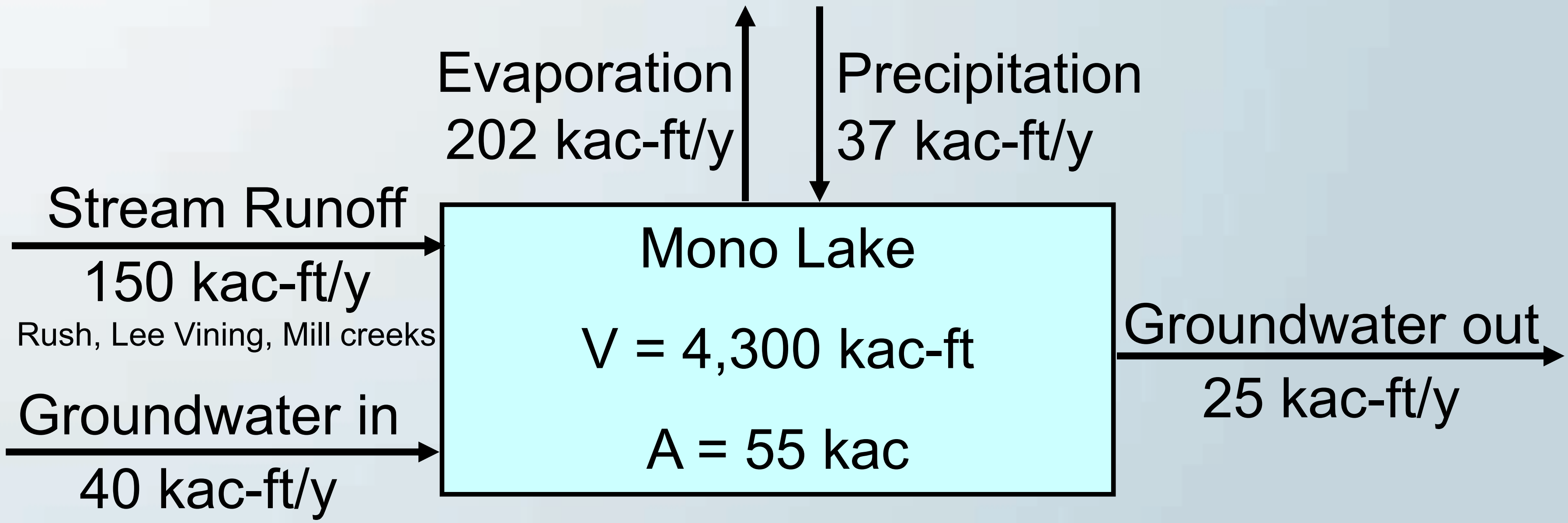
$$\frac{?\ \text{kac}\bullet\text{ft precip}}{1\ \text{yr}} = 55\ \text{kac}\left(\frac{8\ \text{in.}}{1\ \text{yr}}\right)\left(\frac{1\ \text{ft}}{12\ \text{in.}}\right) = 37\ \text{kac}\bullet\text{ft/yr}$$

F_{in} = Stream + Precip + Ground_{in} = 150 + 40 + 37 = 227 kac-ft/yr

In equilibrium, $F_{in} = F_{out}$ = Evap + Ground_{out}

Evap = F_{in} - Ground_{out} = 227 - 25 = 202 kac-ft/y

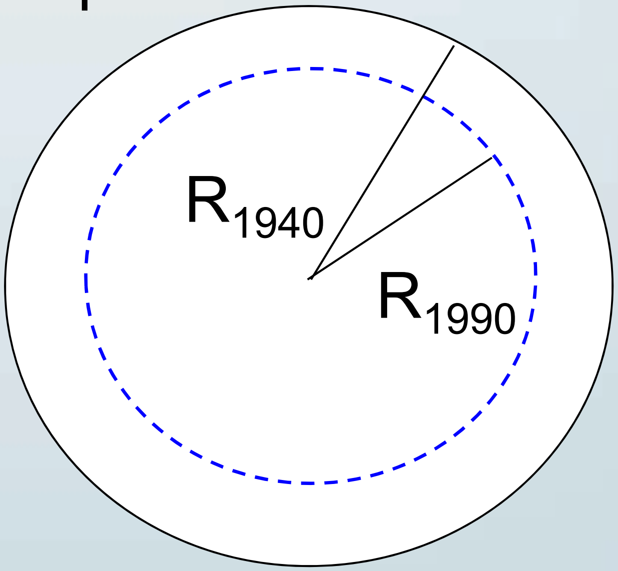
Ex. Using Box Model to Calculate Evaporation from Mono Lake



Equilibrium box model with flows in balance ca 1940.

Changes in Mono Lake

top view of lake



Calculate the 1940 average radius of Mono Lake in feet. (The 1940 area of Mono Lake was 55 kiloacres. There are 4840 yd^2 per acre.)

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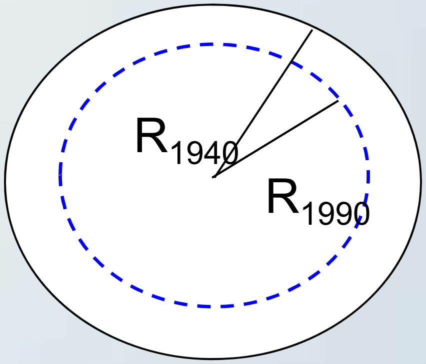

$$A_{1940} = ?\ ft^2 = 55\ \cancel{kacre}\left(\frac{10^3\ \cancel{acre}}{1\ \cancel{kacre}}\right)\left(\frac{4840\ \cancel{yd^2}}{1\ \cancel{acre}}\right)\left(\frac{3\ ft}{1\ \cancel{yd}}\right)^2$$

$$= 2.4 \times 10^9\ ft^2$$

$$A = \pi r^2$$

$$r = \left(\frac{A}{\pi}\right)^{1/2} = \left(\frac{2.4 \times 10^9\ ft^2}{\pi}\right)^{1/2} = 2.8 \times 10^4\ ft$$

What was change in area of lake?

If the surface of the lake dropped 50 ft from 1940 to 1990, calculate Mono Lake's change in radius, its 1990 surface area in Gft^2, and the change in area between 1940 and 1990.

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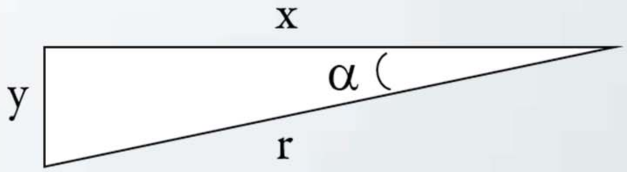
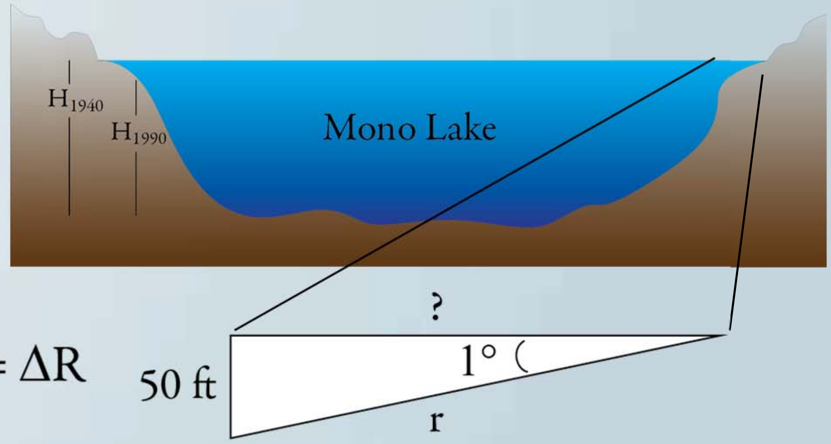




r = shoreline

y = decrease in surface level = 50 ft

x = change in radius of lake when level drops = ΔR

α = angle of shoreline (assume 1°)

$$\tan\alpha = \frac{\text{opposite}}{\text{adjacent}} = \frac{y}{x}$$

$$x = \frac{y}{\tan\alpha} = \frac{50\text{ ft}}{\tan 1^\circ} = \frac{50\text{ ft}}{0.0175} = 2900\text{ ft } (2.4 \times 10^3\text{ ft})$$

$$R_{1990} = R_{1940} - \Delta R = 28{,}000\text{ ft} - 2900\text{ ft} = 25{,}100\text{ ft } (2.5 \times 10^4\text{ ft})$$

$$A_{1990} = \pi R_{1990}{}^2 = \pi(25{,}100\text{ ft})^2\left(\frac{1\text{ Gft}^2}{10^9\text{ ft}^2}\right) = 2.0\text{ Gft}^2$$

$$\Delta A = A_{1940} - A_{1990} = 2.4\text{ Gft}^2 - 2.0\text{ Gft}^2 = 0.4\text{ Gft}^2$$

Box Models for Different Situations

- What is the basic math behind box models?
- What kinds of box model are frequently used in environmental science?
- How can the following situations be analyzed mathematically with box models?
  - population growth
  - radioactive decay
  - greenhouse gas emissions
  - depletion of oil stocks
  - pollution building up in a lake

General Solution to Box Models

1. Draw box model, label stocks and flows
2. Set up differential equation for each box expressing the rate of change of the stock

$$\frac{dS}{dt} = F_{in} - F_{out}$$

3. Solve for S(t) by integrating the differential equation, or in lieu of integrating, use the pre-integrated solutions given in class for S(t). The trick here is to recognize which type of box model calls for which type of solution.

Equations for Different Situations

- Steady State

  $$S(t) = S_0$$

- $F_{in} - F_{out} = \text{constant}$

  $$S(t) = S_0 + \Delta Ft$$

- Exponential Growth of Stocks

  $$S(t) = S_0 e^{rt}$$

- Exponential decline (decay) of Stocks

  $$S(t) = S_0 e^{-rt}$$

Equations for Different Situations

- One flow constant and the other proportional to stock (e.g. constant inflow, and an outflow proportional to stock)

$$S(t) = \frac{(rS_0 - F_0)}{r} e^{-rt} + \frac{F_0}{r}$$

- Exponential increase in inflow

$$S(t) = S_0 + \frac{F_0}{r}(e^{rt} - 1)$$

- Exponential increase in outflow

$$S(t) = S_0 - \frac{F_0}{r}(e^{rt} - 1)$$

Case A: Stock is in Steady State

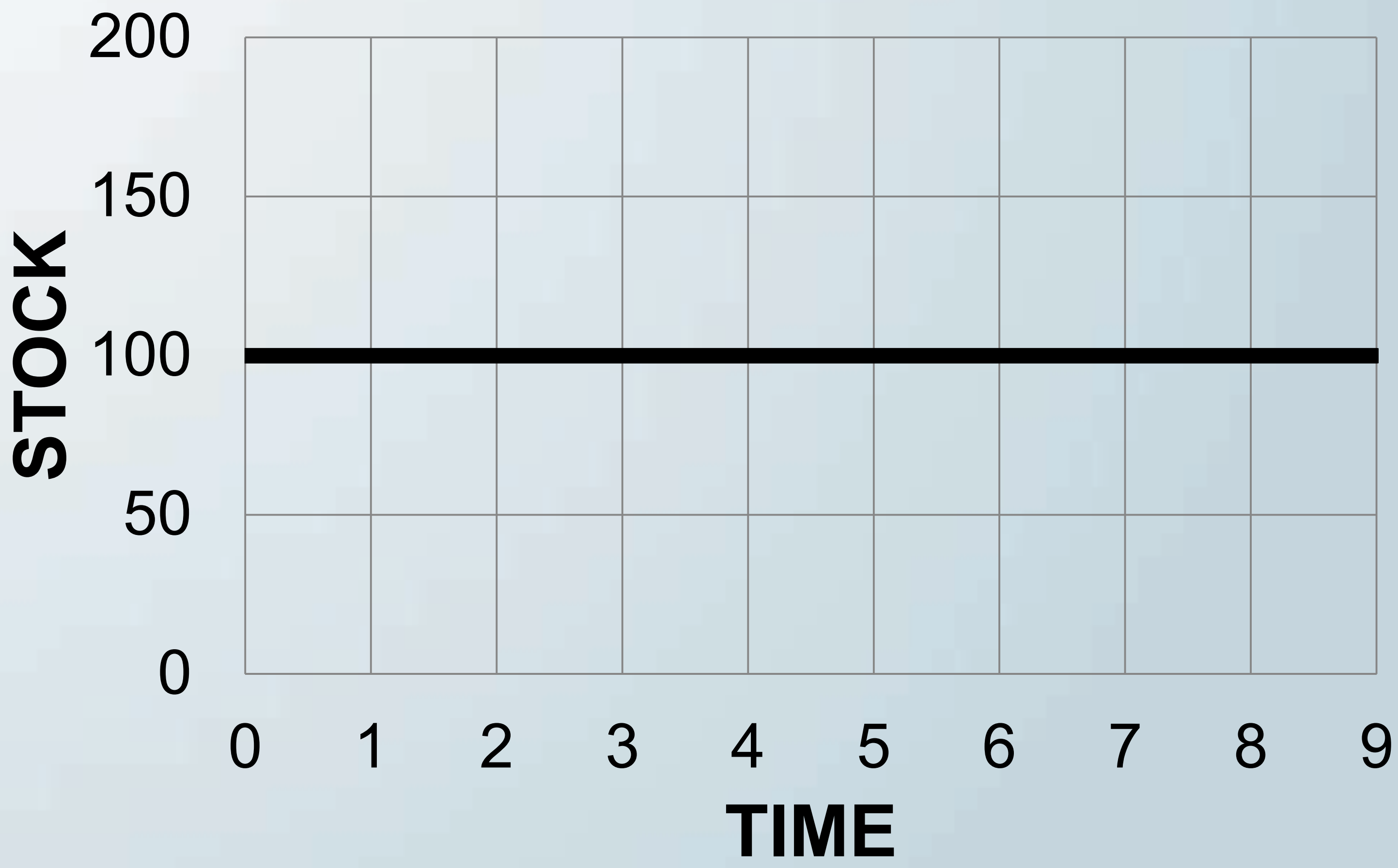
$F_{in} = F_{out} = F$

$S(t) = S_0$ (i.e. no change from initial value)

Residence time T is defined for steady state:

$T = S/F$

Stock is in Steady State: $S(t) = S_0$



Case B: F_{in} - F_{out} = constant

- If the stock is not in a steady state, our calculations involve different equations.
- The simplest case is when the difference between F_{in} and F_{out} is constant:

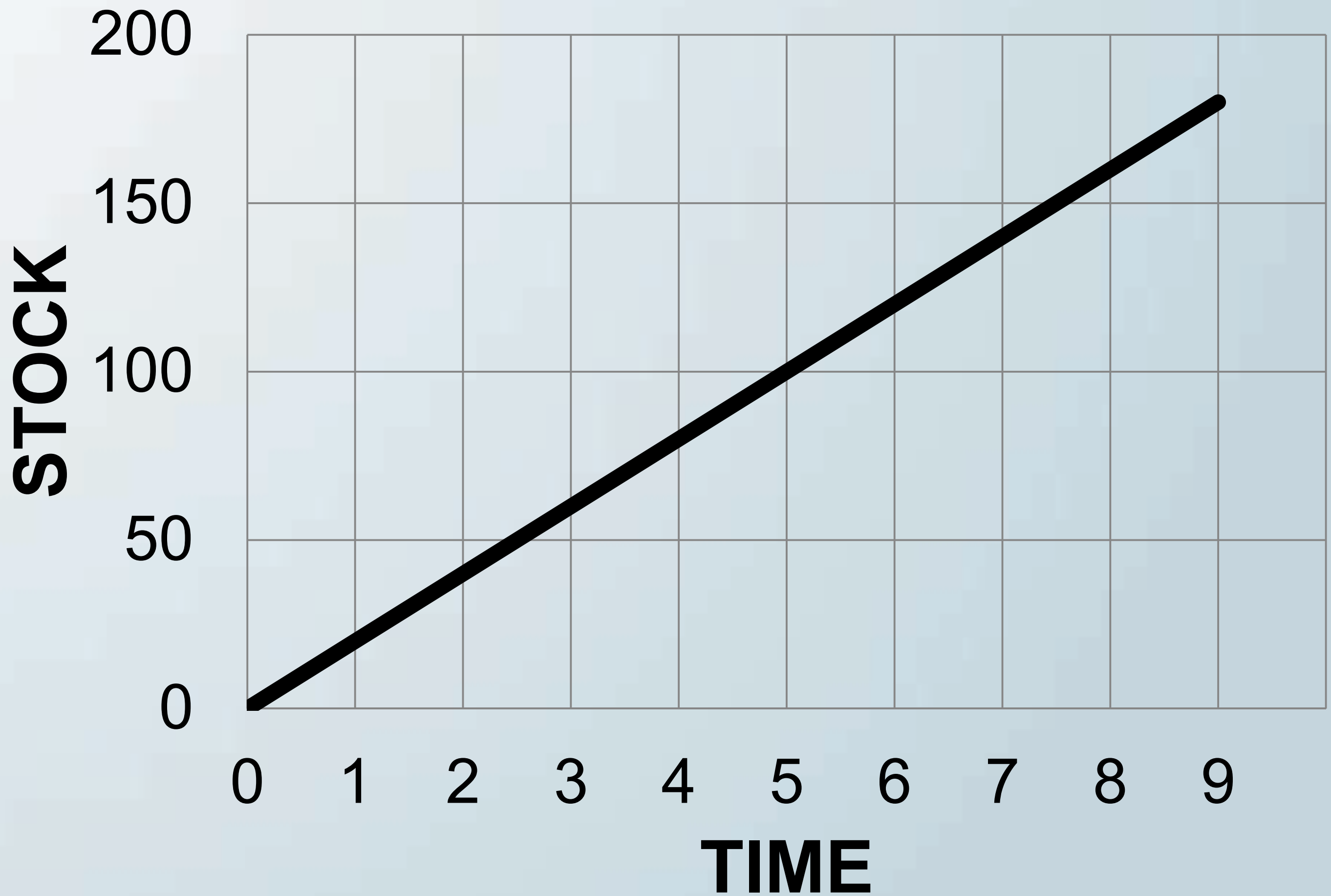
$$\Delta F = F_{in} - F_{out}$$

$$\mathbf{S(t) = S_0 + \Delta F t}$$

Case B: Calculation

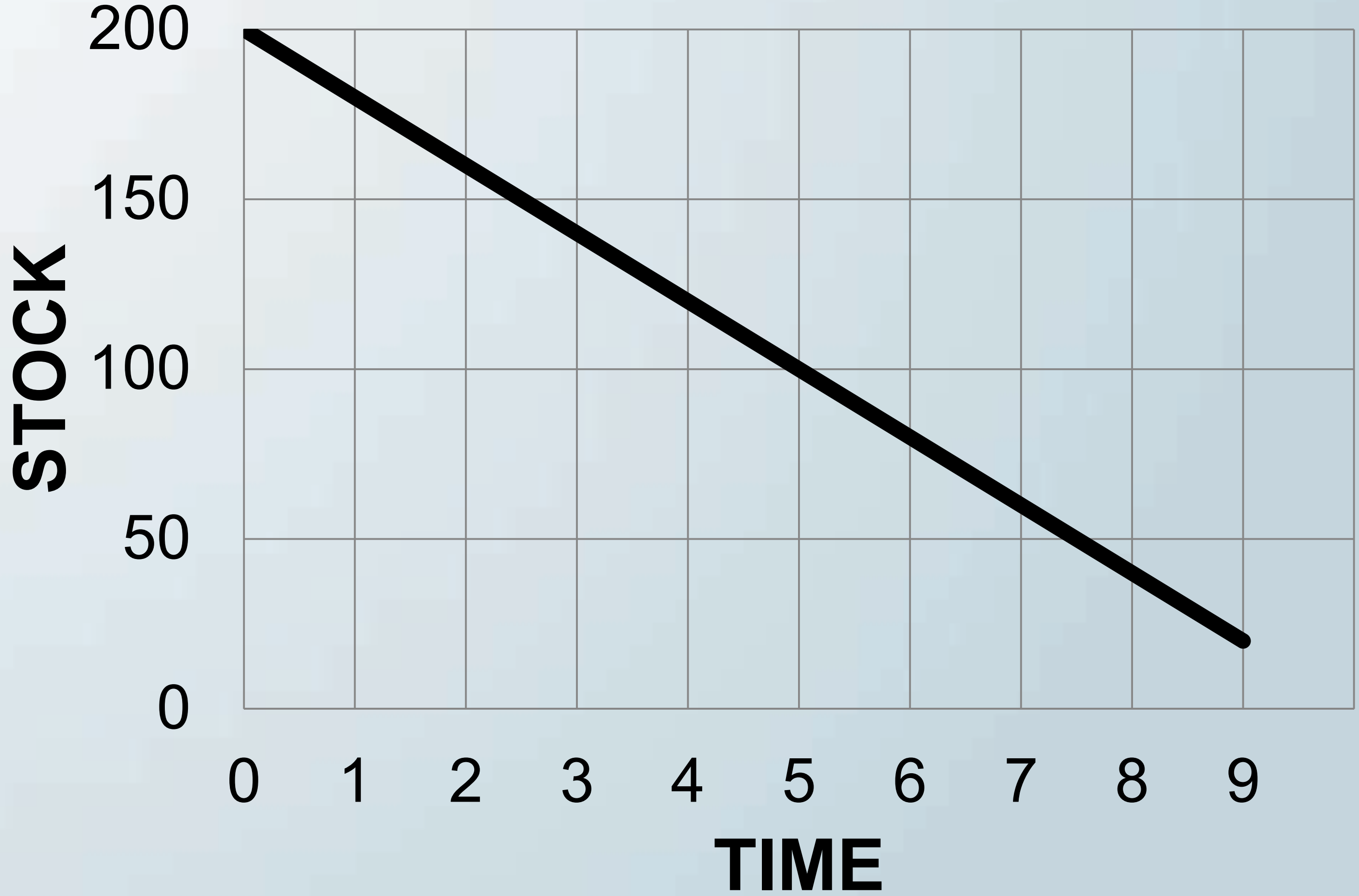
- Consider a water storage tank that you use to irrigate your vineyard. It is 20 ft high and 20 ft in diameter and has a maximum capacity of 4.7×10^4 gallons of water. When you check the water level, you find that it contains 1.2×10^4 gallons, so you decide to fill it up. You can add 80 gal/min, but you know that you will be drawing out about 60 gal/min to irrigate your grape vines. If you start the addition of water now and come back the next day at the same time, will your tank have overflowed?

F_{in} - F_{out} = constant: $S(t) = S_0 + \Delta F t$



Ex. $S_0 = 0$, $\Delta F = 20$

$F_{in} - F_{out}$ = constant: $S(t) = S_0 + \Delta F t$



Ex. $S_0 = 200$, $\Delta F = -20$

Exponential Growth – Fixed Percentage per Year

- Exponential growth when the increase in some quantity is proportional to the amount currently present.
- If fixed percentage per year.

$N_1 = N_0 + rN_0 = N_0(1 + r)$

$N_2 = N_1(1 + r) = N_0(1 + r)(1 + r)$

$N_3 = N_2(1 + r) = N_0(1 + r)(1 + r)(1 + r)$ etc.

or $\boldsymbol{N(t) = N_0(1 + r)^t}$

$N(t)$ = amount at time t

N_0 = initial amount

r = rate

t = time

Example

- If you borrow $100 at a 10% fixed rate, and if you repay the loan in four years, how much will you owe?

  $N(t) = N_0(1 + r)^t$

  $N(t) = \$100(1 + 0.10)^4$

  $N(t) = \$146$

Exponential Growth – Smooth, Continuous

- If we assume that the rate of change is smooth and continuous, the equation is

$$N(t) = N_0e^{rt}$$

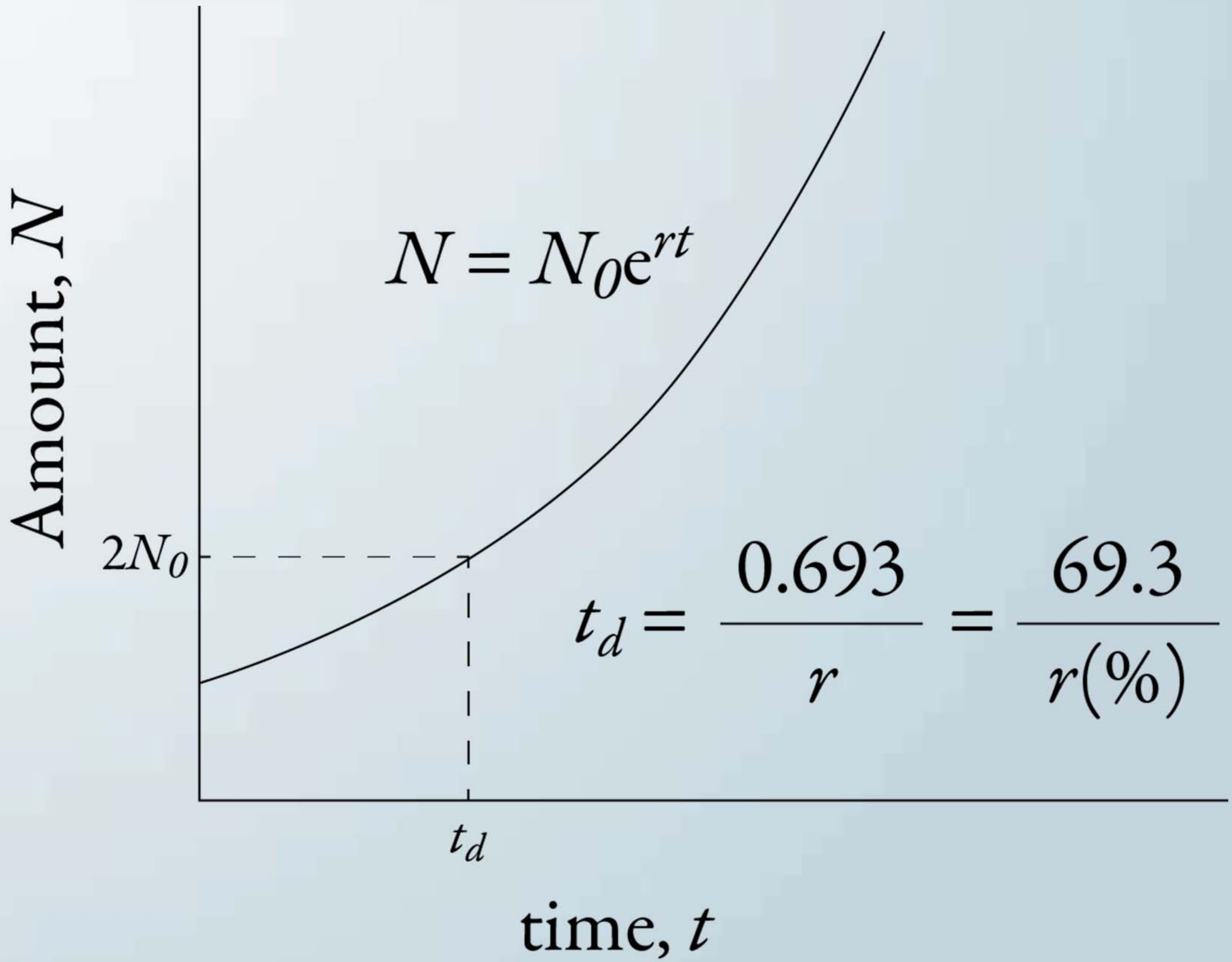
$N(t)$ = amount at time t

N_0 = initial amount

r = rate

t = time

Continuous Exponential Growth



Logarithms

- The logarithm of a number is the exponent by which a fixed number, the base, has to be raised to produce that number. The log base *a* of a number *y* is the power of *a* that yields *y*.

$\log_a y = \log_a a^x = x$ e.g $\log_{10} 1000 = \log_{10} 10^3 = 3$

$\log_{10}$ is commonly described as just log.

Log_e is called the natural log and is commonly described as ln.

e = 2.7182818284590452353602874713526624977572...

$\log_a a = 1$ e.g. $\log 10 = \log 10^1 = 1$ or $\ln e = \ln e^1 = 1$

$\text{Log}_a (b \bullet c) = \log_a b + \log_a c$

$\text{Log}_a (b \div c) = \log_a b - \log_a c$

$\text{Log}_a b^c = c \log_a b$ e.g. $\ln 2^{-3} = -3 \ln 2$

Doubling Time

$N(t)$ = amount time t N_0 = amount at time 0 t = time

r = rate Rate has units of 1/time,

e.g. a rate of change of 12% per year corresponds to 0.12/yr

$N(t) = N_0 e^{rt}$

for doubling $N(t) = 2N_0$

so $2N_0 = N_0 e^{rt_d}$

$2 = e^{rt_d}$

$\ln 2 = \ln (e^{rt_d})$

$\ln 2 = rt_d \ln e$

$\ln 2 = rt_d$

$\frac{\ln 2}{r} = t_d = \frac{\mathbf{0.693}}{r}$

Case C: Exponential Growth of Stocks

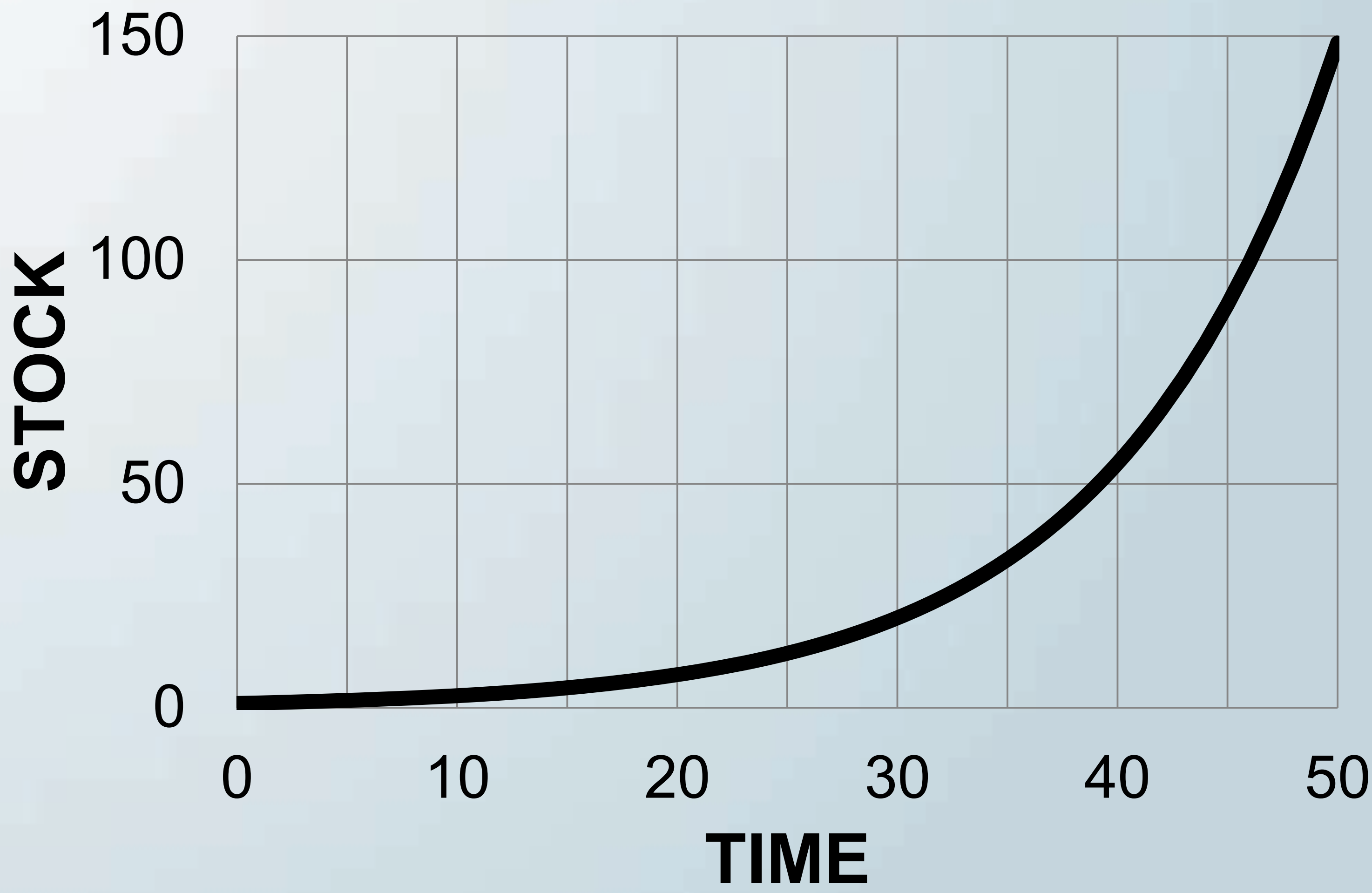
- If you put $100 in an account with a 0.84% annual percentage rate, how much money will you have after 10 years?

$$S(t) = S_0 e^{rt} = \$100\ e^{(0.0084\ 1/\mathrm{yr}\ 10\ \mathrm{yr})} = \$108.76$$

- How long will it take to double your money?

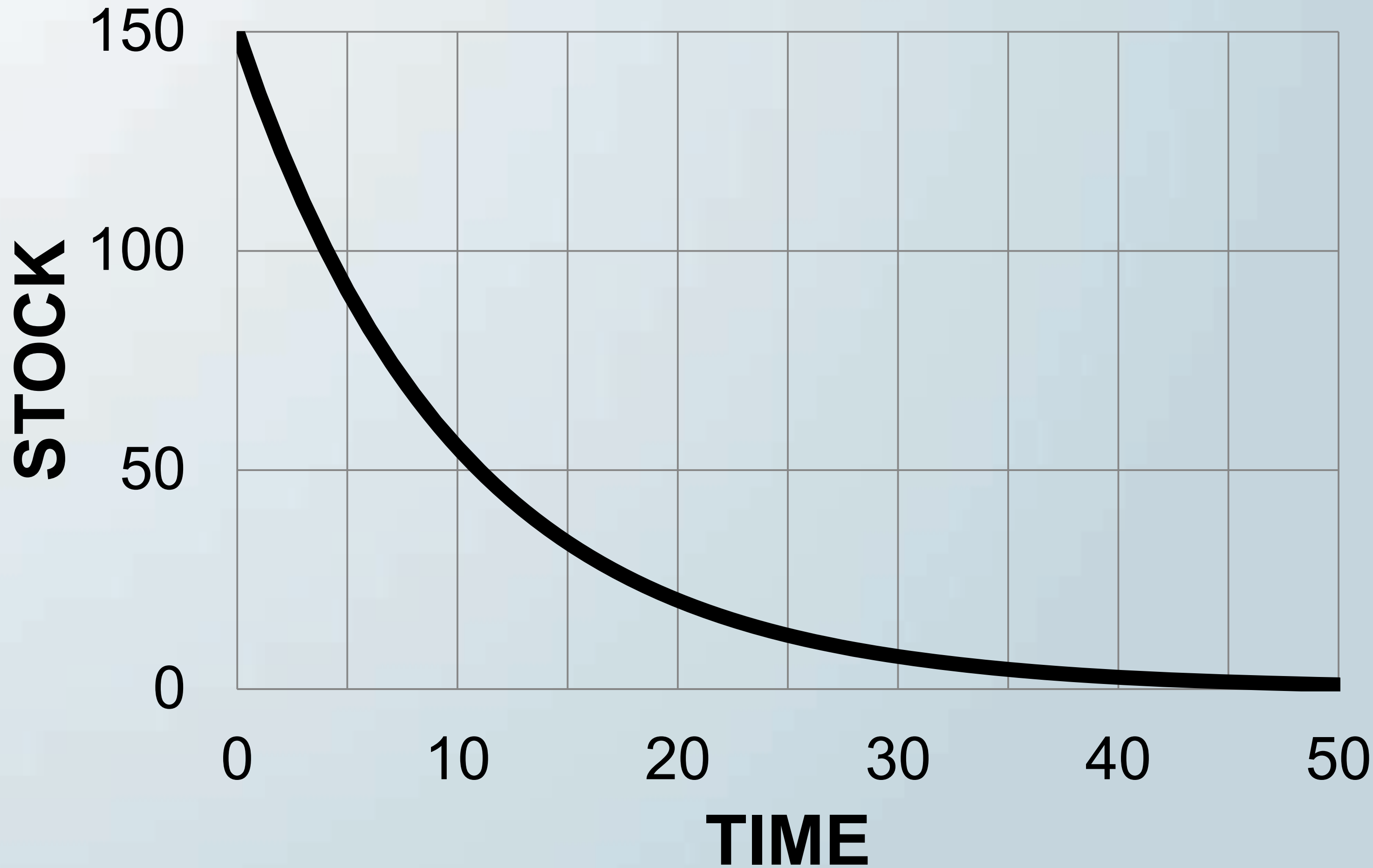
$$T_{1/2} = 0.693/r = 0.693/(0.0084\ 1/y) = 82\ \text{years}$$

Exponential Growth of Stocks: $S(t) = S_0 e^{rt}$



Ex. S_0=1, r=0.1/y

Exponential Decline of Stocks: $S(t) = S_0e^{-rt}$



Ex. S_0=150, r=0.1/y

Exponential Decline and Half-Life

$N(t)$ = amount time t $\quad N_0$ = amount at time 0 $\quad t$ = time
k = rate $\quad k$ has units of 1/time,

$$N(t) = N_0 e^{-kt}$$

When half gone $N(t) = 1/2N_0$

$$\text{so } 1/2N_0 = N_0 e^{-kt_{1/2}}$$

$$2^{-1} = e^{-kt_{1/2}}$$

$$\ln 2^{-1} = \ln(e^{-kt_{1/2}})$$

$$-\ln 2 = -kt_{1/2} \ln e$$

$$\ln 2 = kt_{1/2}$$

$$\frac{\ln 2}{k} = t_{1/2} = \frac{\mathbf{0.693}}{k}$$

Exponential Decline of Stocks Example

- Iodine-131 has a half-life of 8.0197 days. If we start with 37 GBq (or 1 curie) of I-131, how much is left after 14 days?
  - The **becquerel** (symbol **Bq**) (pronounced: 'be-kə-rel) is the SI-derived unit of radioactivity. One Bq is defined as the activity of a quantity of radioactive material in which one nucleus decays per second. The Bq unit is therefore equivalent to an inverse second, s^{-1}.
  - The curie is a common non-SI unit. It is now defined as 37 GBq.

Iodine-131 has a half-life of 8.0197 days. If we start with 37 GBq (or 1 curie) of I-131, how much is left after 14 days?

$$t_{1/2} = \frac{0.693}{k} \qquad k = \frac{0.693}{t_{1/2}} = \frac{0.693}{8.0197 \text{ day}} = 0.0864 \text{ day}^{-1}$$

$$S(t) = S_o\, e^{-kt} = 37 \text{ GBq}\, e^{-0.0864\ 1/\text{day}(14 \text{ day})} = \mathbf{11\ GBq}$$

Measuring CO_2 from fossil fuels by Measuring $^{14}CO_2/^{12}CO_2$ Ratio

- About one in a trillion CO_2 molecules naturally contain ^{14}C, but the carbon locked up in fossil fuels, such as coal and oil has none.
- Because the half-life of ^{14}C is 5730 years, the carbon-14 in fossil fuels decayed to ^{14}N millions of years ago

$$^{14}_{6}C \rightarrow {}^{14}_{7}N + {}^{0}_{-1}e$$

- Therefore, as the amount of CO_2 in the atmosphere from fossil fuels rises, the ratio of $^{14}CO_2/^{12}CO_2$ decreases in a measurable way.

Measuring CO_2 from fossil fuels by Measuring $^{14}CO_2/^{12}CO_2$ Ratio

- Monitoring stations in the Swiss Alps and in Antarctica found that the $^{14}CO_2/^{12}CO_2$ is now about 0.5% lower in the Northern Hemisphere than in the Southern
- Reversal since preindustrial days…tree ring records show that the Southern Hemisphere had less $^{14}CO_2$ because upwelling of deep waters in the southern oceans brought radiocarbon-depleted CO_2 to the surface.

Measuring CO_2 from fossil fuels by Measuring $^{14}CO_2/^{12}CO_2$ Ratio

- Uncertainties in estimates of fossil-fuel emissions from $^{14}CO_2$ are still large, because accurate modeling depends on knowing all possible sources of both radioactive and nonradioactive CO_2.
- One confounding factor is emissions from nuclear power plants, which generate a significant amount of ^{14}C in regions where nuclear plants are concentrated, offsetting at least 20% of the reduction in $^{14}CO_2$ due to fossil fuels.

Case D. One Flow Constant, One Flow is Proportional to Stock

If there is a constant inflow, and an outflow proportional to stock:

$$\frac{dS}{dt} = F_0 - rS \rightarrow \frac{dS}{F_0 - rS} = dt \rightarrow \frac{-1}{r}\frac{d(F_0 - rS)}{F_0 - rS} = dt$$

$$\int_{S_0}^{S} \frac{d(F_0 - rS)}{F_0 - rS} = -r\int_0^t dt \rightarrow \ln\left(\frac{F_0 - rS}{F_0 - rS_0}\right) = -rt$$

$$F_0 - rS = (F_0 - rS_0)e^{-rt} \rightarrow -rS = (F_0 - rS_0)e^{-rt} - F_0$$

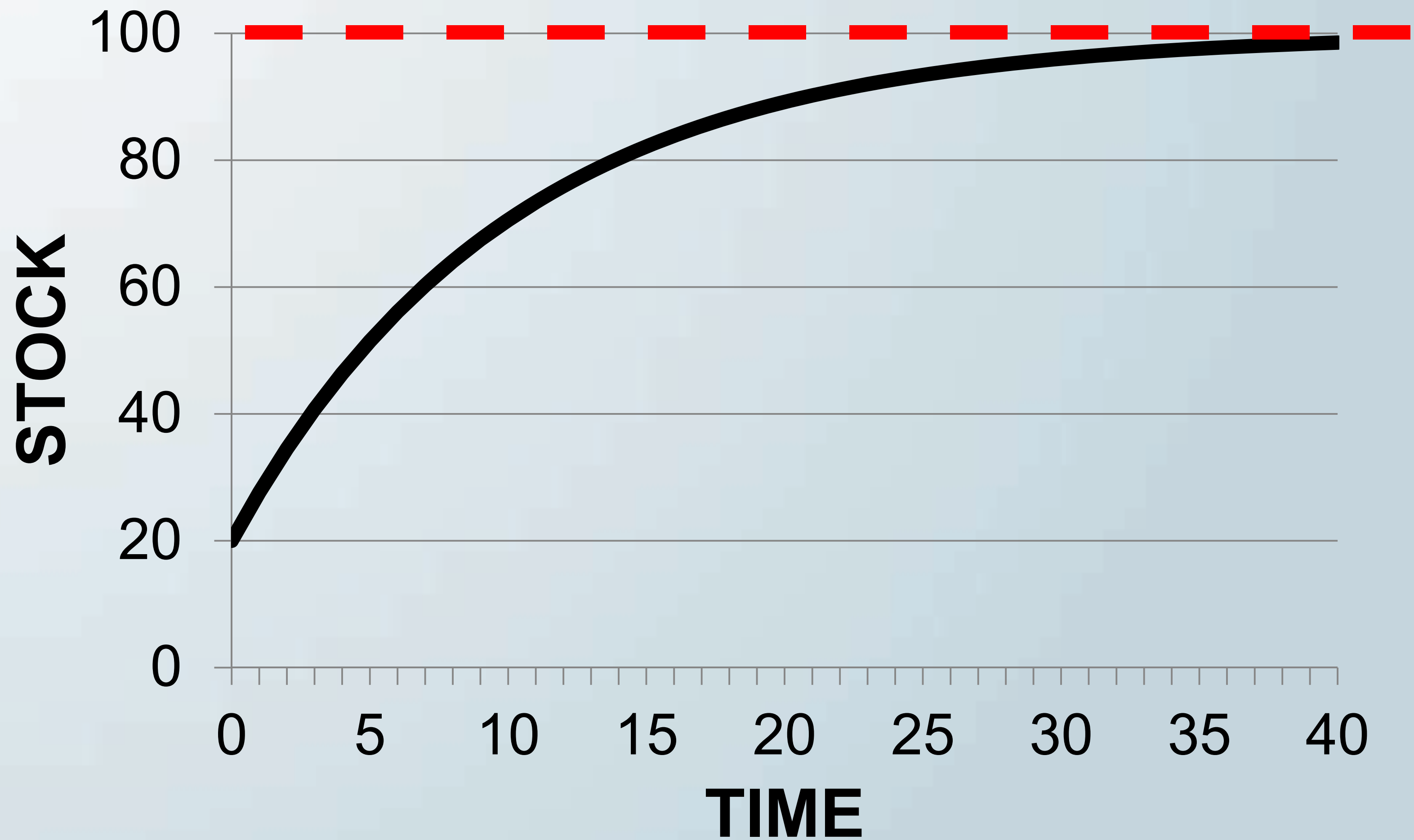
$$S(t) = \frac{(rS_0 - F_0)}{r}e^{-rt} + \frac{F_0}{r}$$

One Flow Constant, One Flow is Proportional to Stock (cont.)

$$S(t) = \frac{(rS_0 - F_0)}{r} e^{-rt} + \frac{F_0}{r}$$

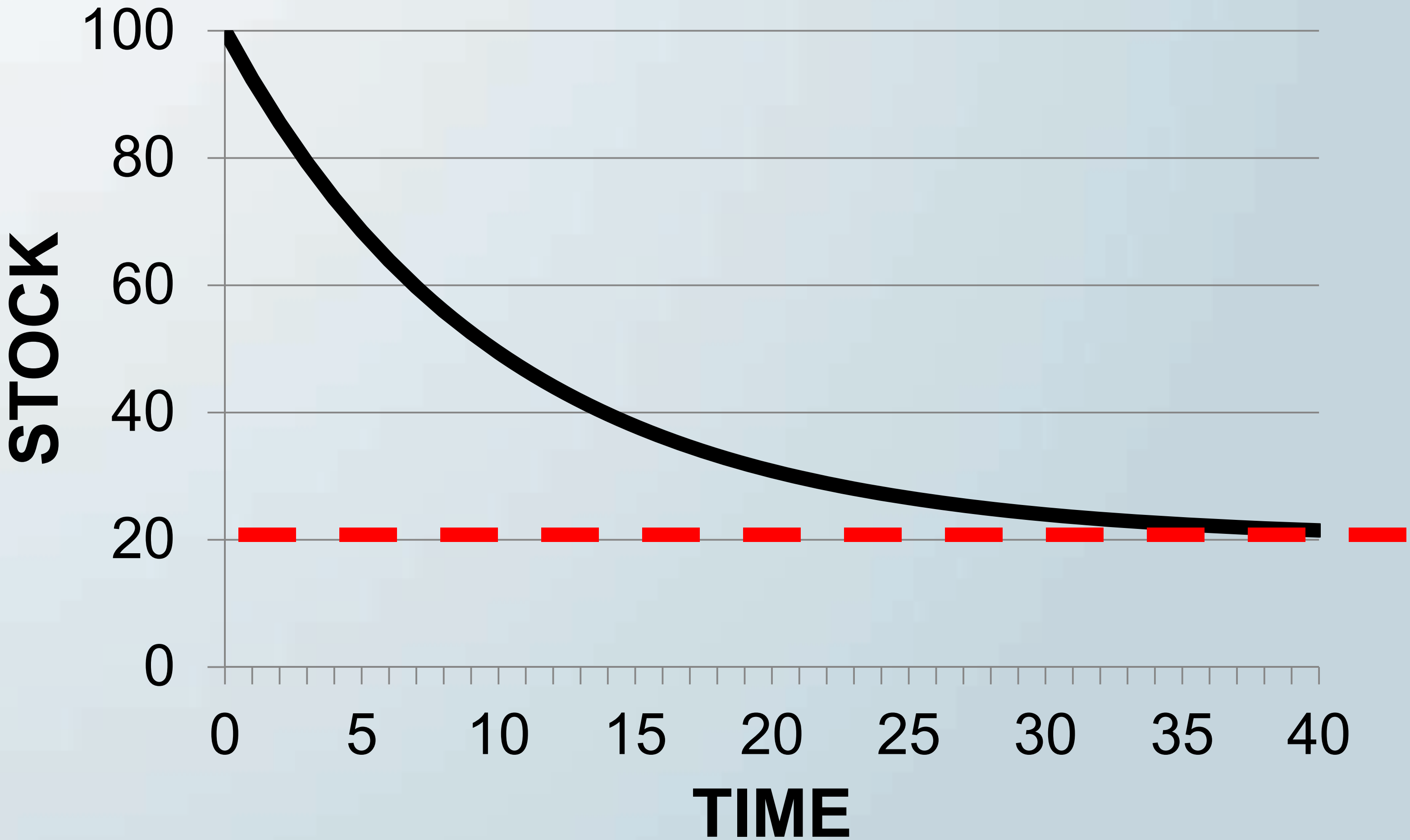
- Note two things about this formula
  (1) $t = 0$, $S(t) = S_0$
  (2) $t = \infty$, $S(t) = F_0/r$
  Thus S approaches F_0/r asymptotically, either from above or below depending on the relative values of S_0 and F_0/r
- This is the solution to problems such as pollution buildup in a lake (see COW 114-115) where pollution flows in at a constant rate, but flows out in proportion to its concentration in the lake (i.e. F_{out} is directly proportional to the stock of pollutant, hence an exponential term)

$$S(t) = \frac{(rS_0 - F_0)}{r} e^{-rt} + \frac{F_0}{r}$$



Ex. S_0=20, F_0=10/y, r=0.1/y

$$S(t) = \frac{(rS_0 - F_0)}{r}e^{-rt} + \frac{F_0}{r}$$



Ex. S_0=100, F_0=2/y, r=0.1/y

Case E. Exponential Growth of Flows

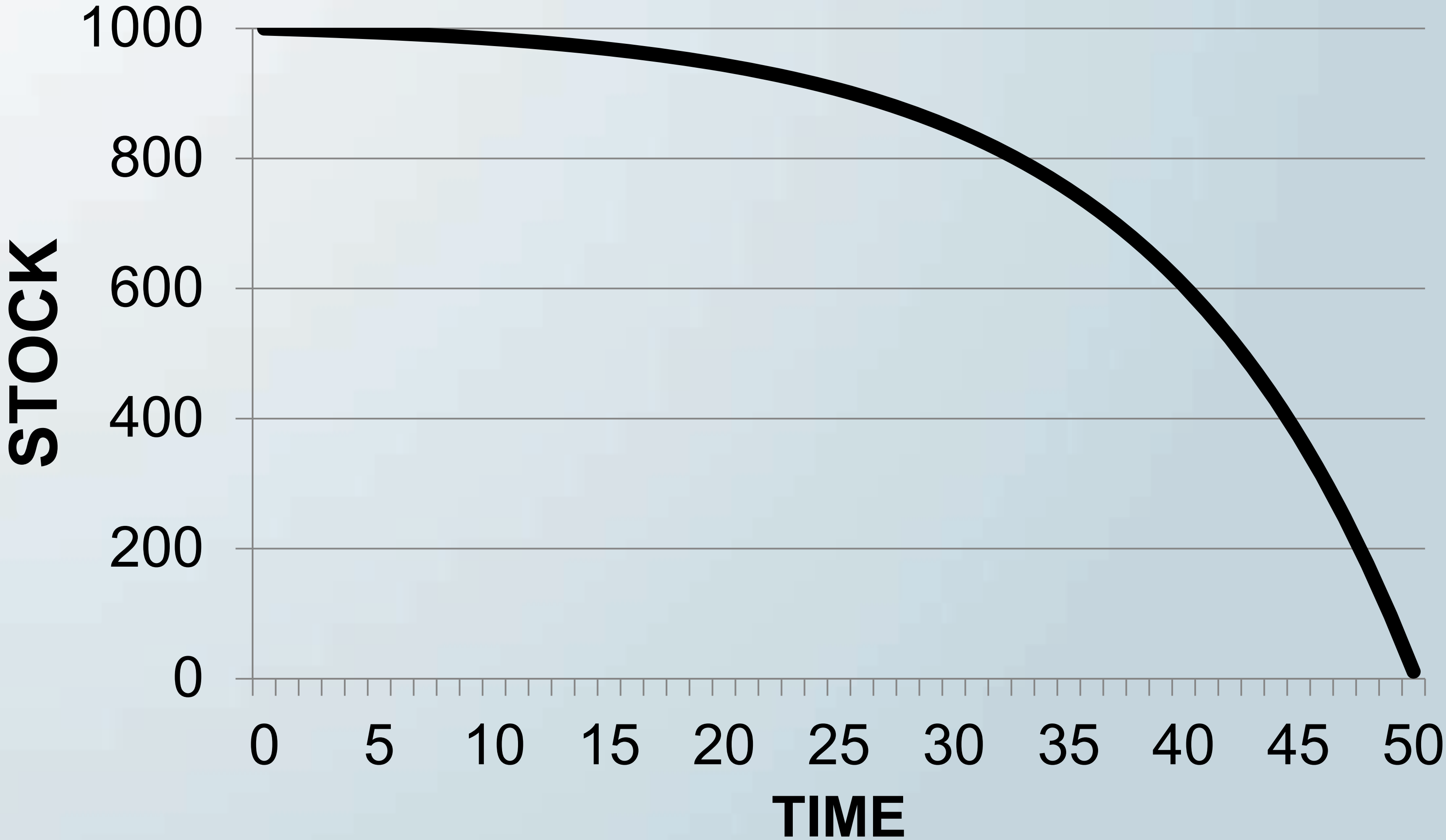
If the inflow increases exponentially:

$$\frac{dS}{dt} = F_0 e^{rt} \rightarrow \boxed{S(t) = S_0 + \frac{F_0}{r}(e^{rt} - 1)}$$

If the outflow increases exponentially:

$$\frac{dS}{dt} = -F_0 e^{rt} \rightarrow \boxed{S(t) = S_0 - \frac{F_0}{r}(e^{rt} - 1)}$$

$$S(t) = S_0 - \frac{F_0}{r}(e^{rt} - 1)$$



Ex. S_0=1000, F_0=1/y, r=0.09/y

Case E (Cont.)

To find the time T it takes for the total stock to be used up, e.g. all remaining oil to be consumed:

$$S(T) = 0 = S_0 - \frac{F_0}{r}(e^{rT} - 1) \rightarrow S_0 = \frac{F_0}{r}(e^{rT} - 1)$$

$$\rightarrow \frac{rS_0}{F_0} = (e^{rT} - 1) \rightarrow e^{rT} = \frac{rS_0}{F_0} + 1 = rT = \ln\left(\frac{rS_0}{F_0} + 1\right)$$

$$T = \frac{1}{r}\ln\left(\frac{rS_0}{F_0} + 1\right)$$

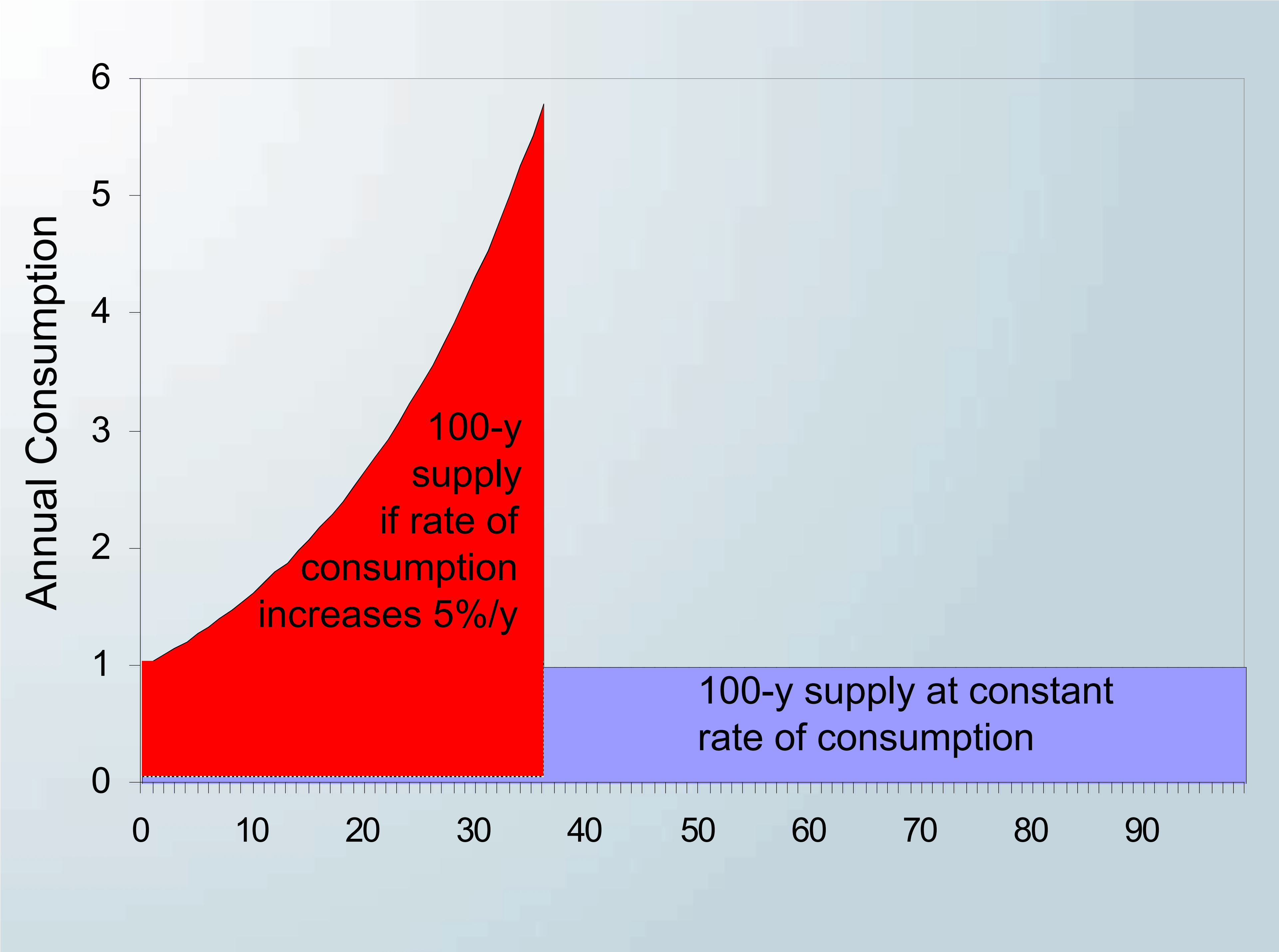
Ex. Exponentially Growing Oil Depletion

- How long to consume a total amount of oil S?

$$T = \frac{1}{r}\ln\left(\frac{rS_0}{F_0} + 1\right)$$

- Suppose we have a 100-y supply of oil at current rate of consumption (S_0/F_0 = 100 y), but consumption is growing 5%/y (r = 0.05/y)

$$T = \left(\frac{y}{0.05}\right)\ln\left[\left(\frac{0.05}{y}\right)(100\ y) + 1\right] = \frac{\ln(6)}{\left(\frac{0.05}{y}\right)} = 36\ y$$


Annual Consumption
6
5
4
3
2
1
0
0
10
20
30
40
50
60
70
80
90
100-y
supply
if rate of
consumption
increases 5%/y
100-y supply at constant
rate of consumption

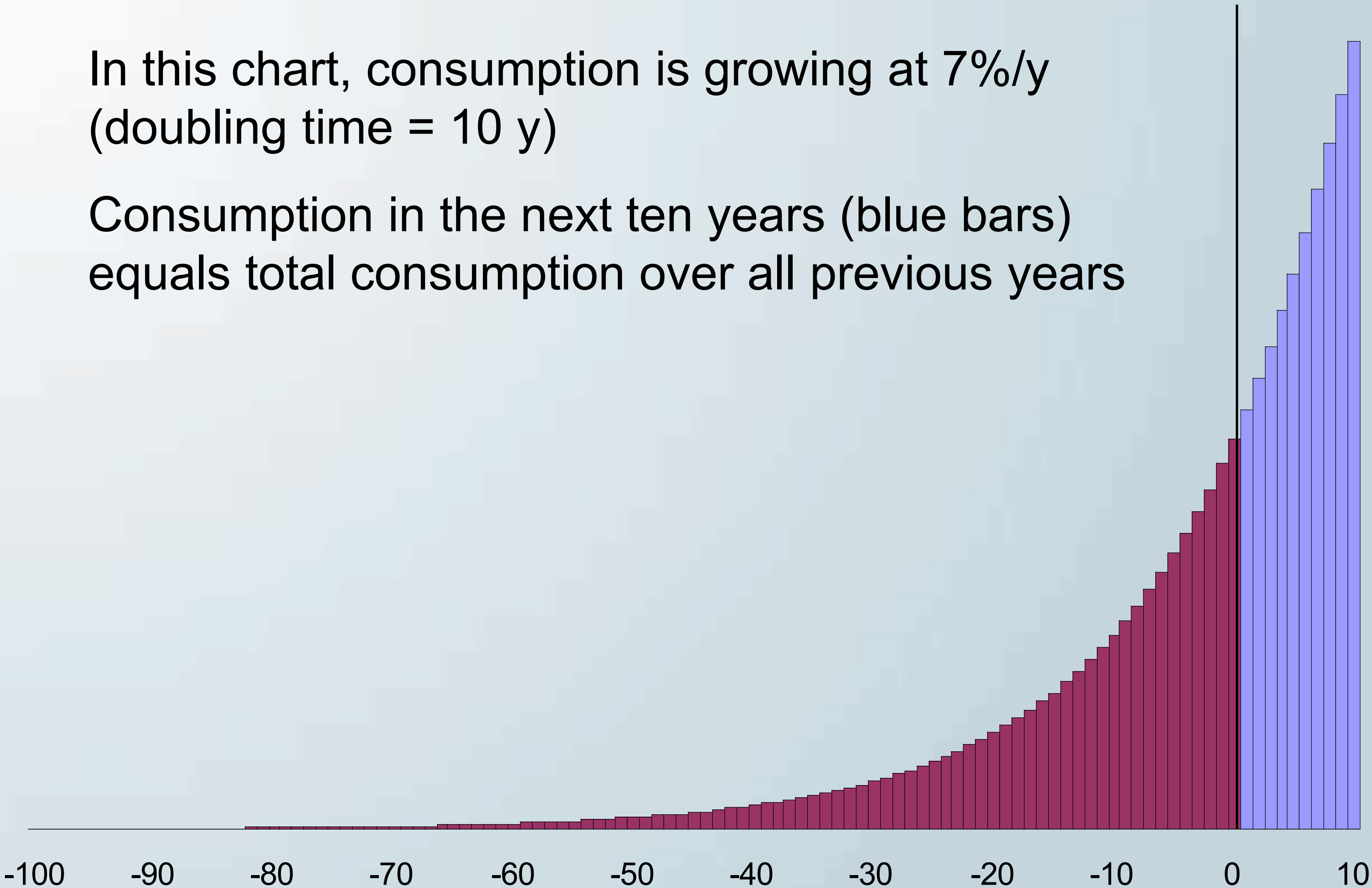
Observation on Exponential Growth of Flows

- If consumption rate is growing exponentially, total consumption in the next doubling time T_{2X} equals all previous consumption:

$$\frac{S\left(0 \text{ to } T_{2X}\right)}{S\left(-\infty \text{ to } 0\right)} = \frac{\frac{F_0}{r}\left[e^{rT_{2X}} - e^{0}\right]}{\frac{F_0}{r}\left[e^{0} - e^{-\infty}\right]} = \frac{e^{r\left(\frac{0.69}{r}\right)} - 1}{1-0} = \frac{2-1}{1} = 1$$

In this chart, consumption is growing at 7%/y (doubling time = 10 y)

Consumption in the next ten years (blue bars) equals total consumption over all previous years



Goals

- To estimate the mass of carbon released in the future
- To clarify how the release of carbon can be minimized

Steps to Meet Goals

- Identify the most important general factors that effect carbon release.
  - Population
  - Affluence
  - Energy efficiency
  - Carbon efficiency
- Attempt to quantify each factor.
  - P = population (pers)
  - GDP/P = per capita economic activity ($/pers)
  - E/GDP = energy intensity of economic activity (GJ/$)
  - C/E = carbon intensity of energy supply (kg/GJ)

POPULATION

History of Earth in one year: Carl Sagan's "Cosmic Calendar"

Jan 1, 12:00 AM: Formation of Earth

Mar 27, 6:30 AM: Origin of life

Jul 17, 4:00 PM: First eukaryotes

Nov 18, 8:15 PM: Cambrian Explosion

Dec 13, 3:50 PM: Dinosaurs appear

Dec 21, 3:10 AM: Flowering plants appear

Dec 25, 5:00 AM: Primates appear

Dec 26, 7:20 PM: Dinosaurs go extinct

Dec 31, 7:12 PM: Genus *Homo* appears

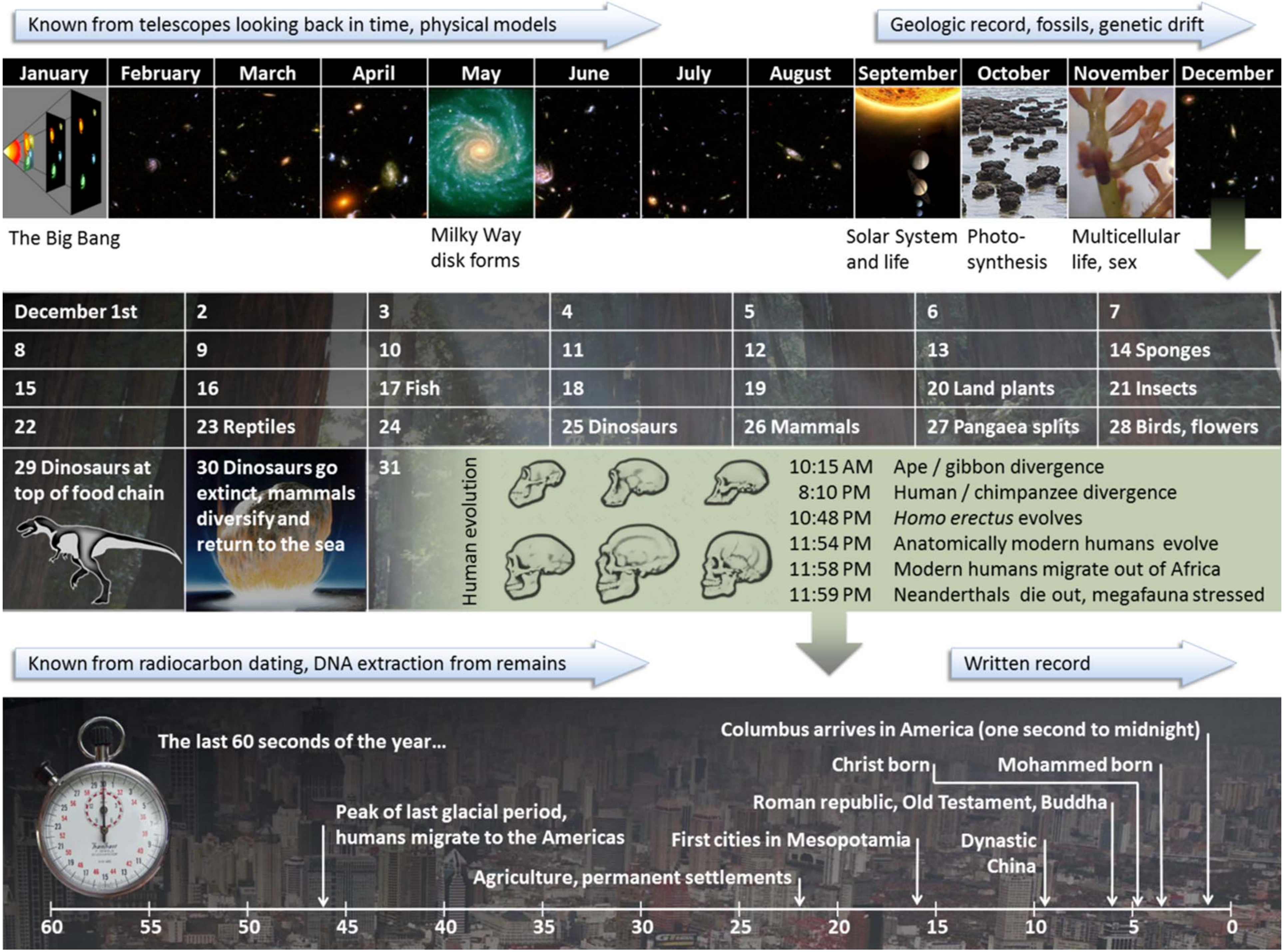
Dec 31, 11:36 PM: *Homo sapiens* appears

Dec 31, 11:58:50 PM: Invention of agriculture

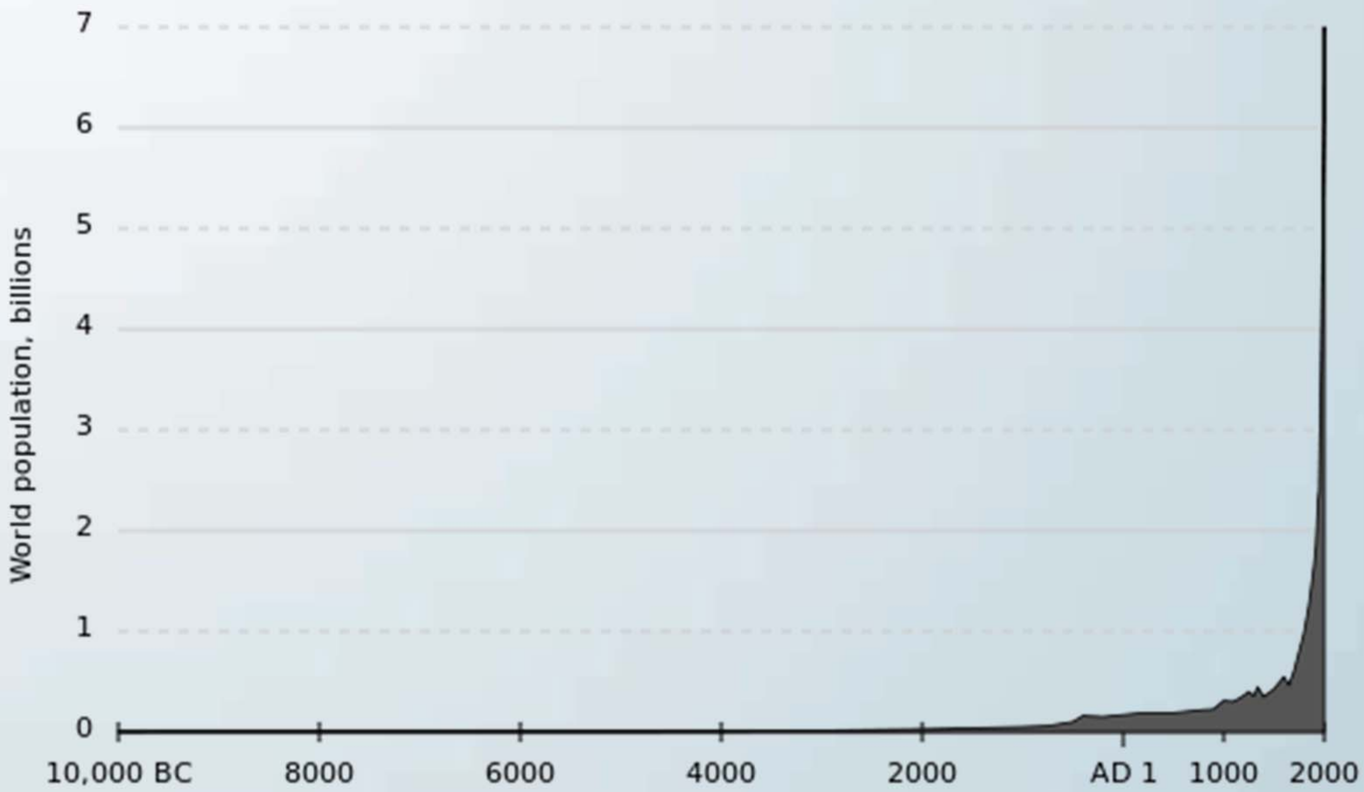
Dec 31, 11:59:23 PM: Invention of writing

Dec 31, 11:59:59 PM: First spaceflight

Humans are very recent arrivals!


Known from telescopes looking back in time, physical models
Geologic record, fossils, genetic drift
January
February
March
April
May
June
July
August
September
October
November
December
The Big Bang
Milky Way disk forms
Solar System and life
Photo-synthesis
Multicellular life, sex
December 1st
2
3
4
5
6
7
8
9
10
11
12
13
14 Sponges
15
16
17 Fish
18
19
20 Land plants
21 Insects
22
23 Reptiles
24
25 Dinosaurs
26 Mammals
27 Pangaea splits
28 Birds, flowers
29 Dinosaurs at top of food chain
30 Dinosaurs go extinct, mammals diversify and return to the sea
31
Human evolution
10:15 AM Ape / gibbon divergence
8:10 PM Human / chimpanzee divergence
10:48 PM *Homo erectus* evolves
11:54 PM Anatomically modern humans evolve
11:58 PM Modern humans migrate out of Africa
11:59 PM Neanderthals die out, megafauna stressed
Known from radiocarbon dating, DNA extraction from remains
Written record
The last 60 seconds of the year...
Columbus arrives in America (one second to midnight)
Christ born
Mohammed born
Roman republic, Old Testament, Buddha
Peak of last glacial period, humans migrate to the Americas
First cities in Mesopotamia
Dynastic China
Agriculture, permanent settlements
60
55
50
45
40
35
30
25
20
15
10
5
0

History of Human Population



Population	
Year	Billion
1800	1
1927	2
1960	3
1974	4
1987	5
1999	6
2011*	7

* UNFPA United Nations Population Fund estimate 31.10.2011

As of October, 2, 2012, the world human population is estimated to be 7,043,107,338 by the United States Census Bureau and over 7 billion by the United Nations.

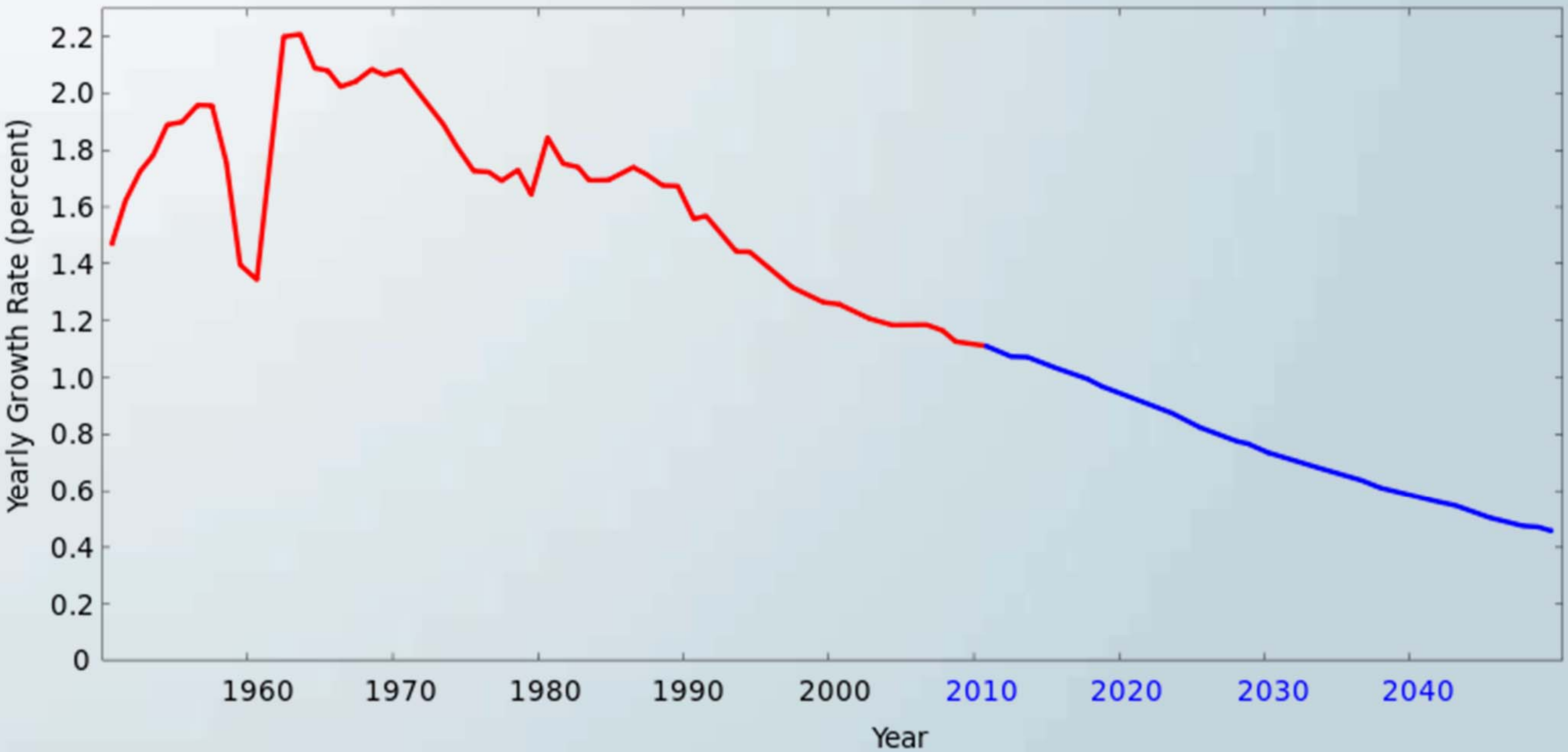
Growth of Human Population

According to United Nations population statistics, the world population grew by 30%, or 1.6 billion people, between 1990 and 2010.

http://esa.un.org/unpd/wpp/unpp/Panel_profiles.htm

Rank	Country	Population 2010	Population 1990	Growth (%) 1990-2010
	World	6,895,889,000	5,306,425,000	30.0%
1	China	1,341,335,000	1,145,195,000	17.1%
2	India	1,224,614,000	873,785,000	40.2%
3	United States	310,384,000	253,339,000	22.5%
4	Indonesia	239,871,000	184,346,000	30.1%
5	Brazil	194,946,000	149,650,000	30.3%
6	Pakistan	173,593,000	111,845,000	55.3%
7	Nigeria	158,423,000	97,552,000	62.4%
8	Bangladesh	148,692,000	105,256,000	41.3%
9	Russia	142,958,000	148,244,000	-3.6%
10	Japan	128,057,000	122,251,000	4.7%

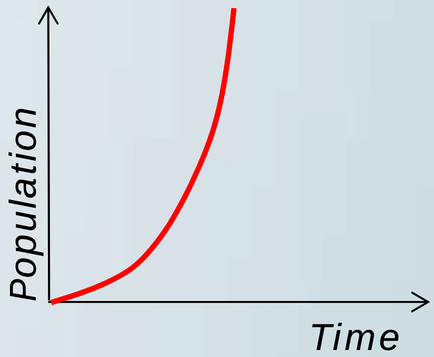
Change in Population Growth Rate



Population Math: Exponential Growth Model

$$S(t) = S_0 e^{rt}$$

$$T_d = 0.693/r$$



- Somalia's population is about 10 million. If their annual birth rate is 5% and their annual death rate is 2%, the net rate of change is 3%. If we *assume that their population growth is exponential*, we can calculate their approximate population in ten year and the doubling time for their population
  - $P_{10} = 10^7\, e^{0.03\ 1/y(10y)} = 13$ million
  - $T_d = 0.693/0.03 = 23$ years
- Problem: Population can't continue to grow exponentially. There are limits to growth.

Carrying Capacity (K)

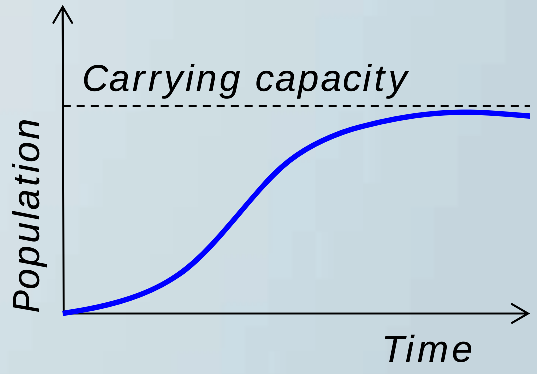
- **Carrying capacity (K)** is the number of individuals an environment can support without significant negative impacts to the given organism and its environment.
- Most estimates for the human carrying capacity of Earth are between 4 billion and 16 billion.

http://www.un.org/esa/population/publications/wpm/wpm2001.pdf

Population Math

Case F: Logistic growth model:

$$S(t) = \frac{KS_0 e^{rt}}{K + S_0(e^{rt} - 1)}$$

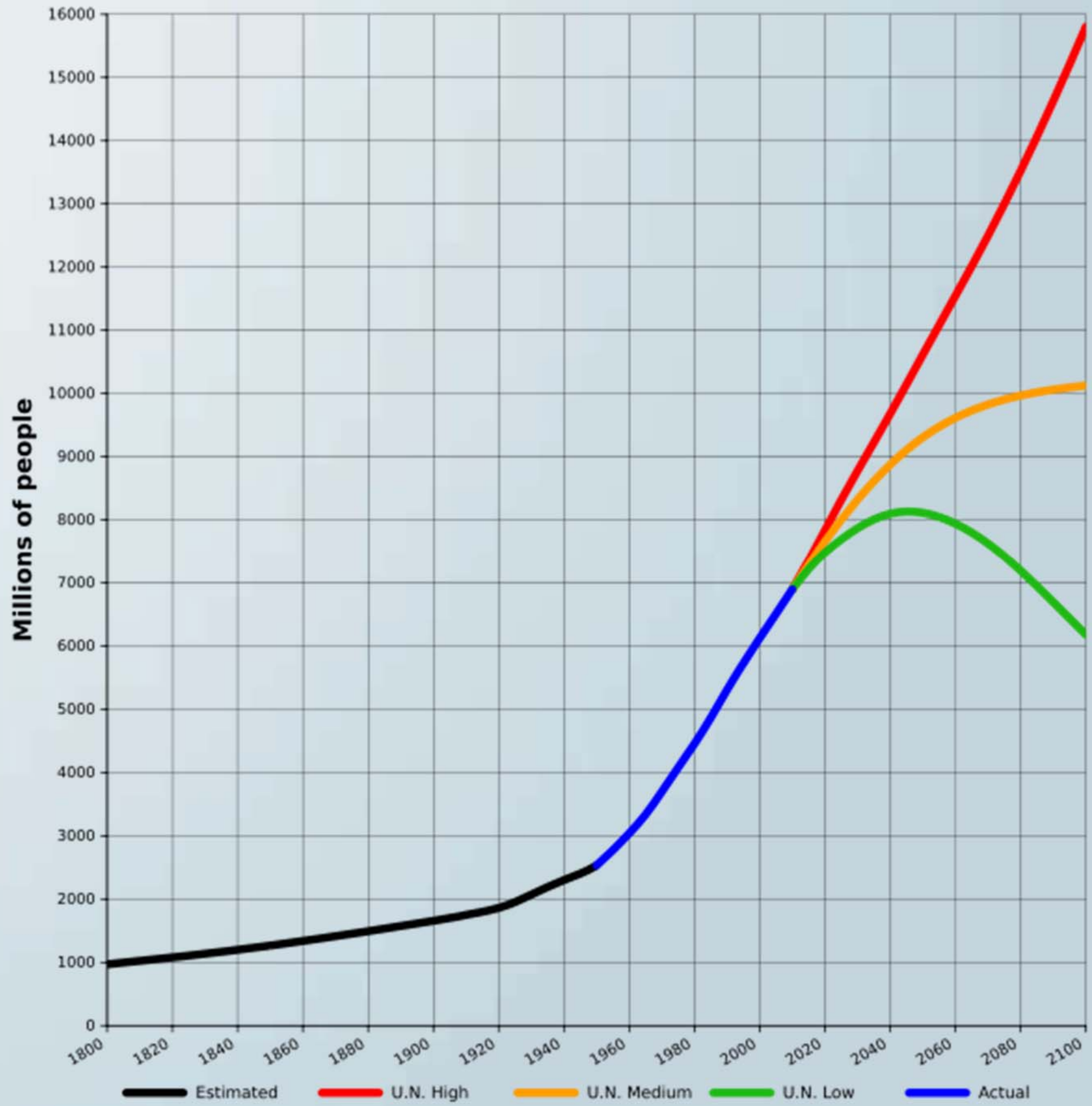


where **S** = number of individuals in population
S_0 = initial size of population
r = net population growth rate (per individual per unit time)
K = carrying capacity (maximum possible population)

Note: Exponential and logistic growth are simple models of population. Actual demographic analysis uses linear algebra (matrix math). See COW, 216-223.

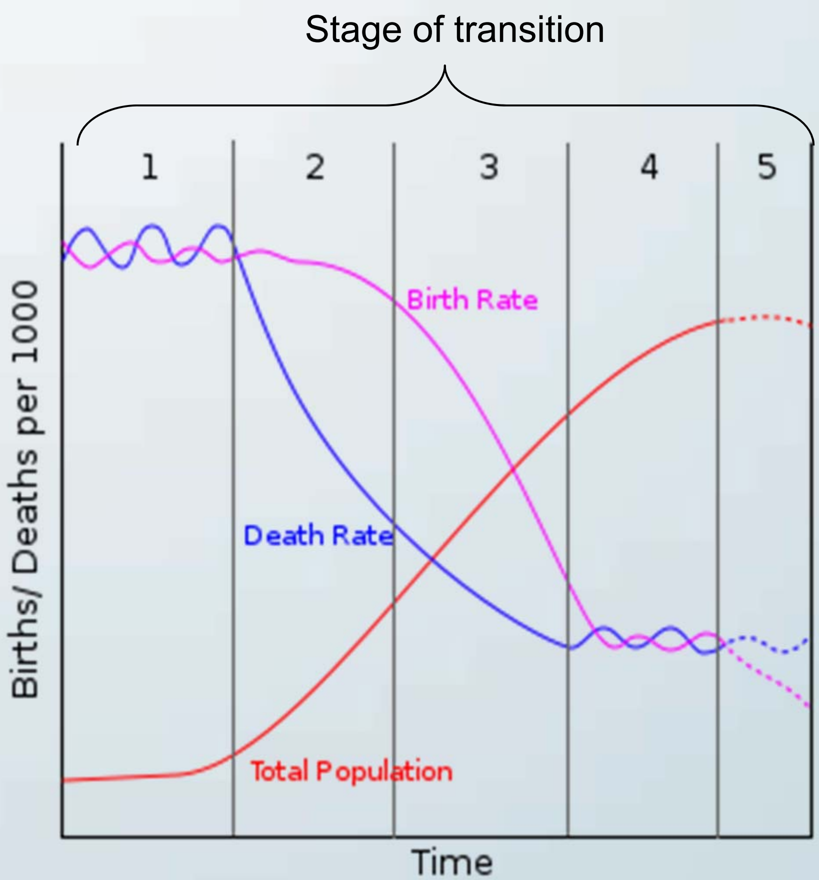
World Population Estimated

- World population estimates from 1800 to 2100, based on UN 2010 projections (**red**, **orange**, **green**) and US Census Bureau historical estimates (**black**). According to the highest estimate, the world population may rise to 16 billion by 2100; according to the lowest estimate, it may decline to only 6 billion.



The demographic transition

Population growth rate = birth rate - death rate



Source: http://en.wikipedia.org

<u>Stages</u>

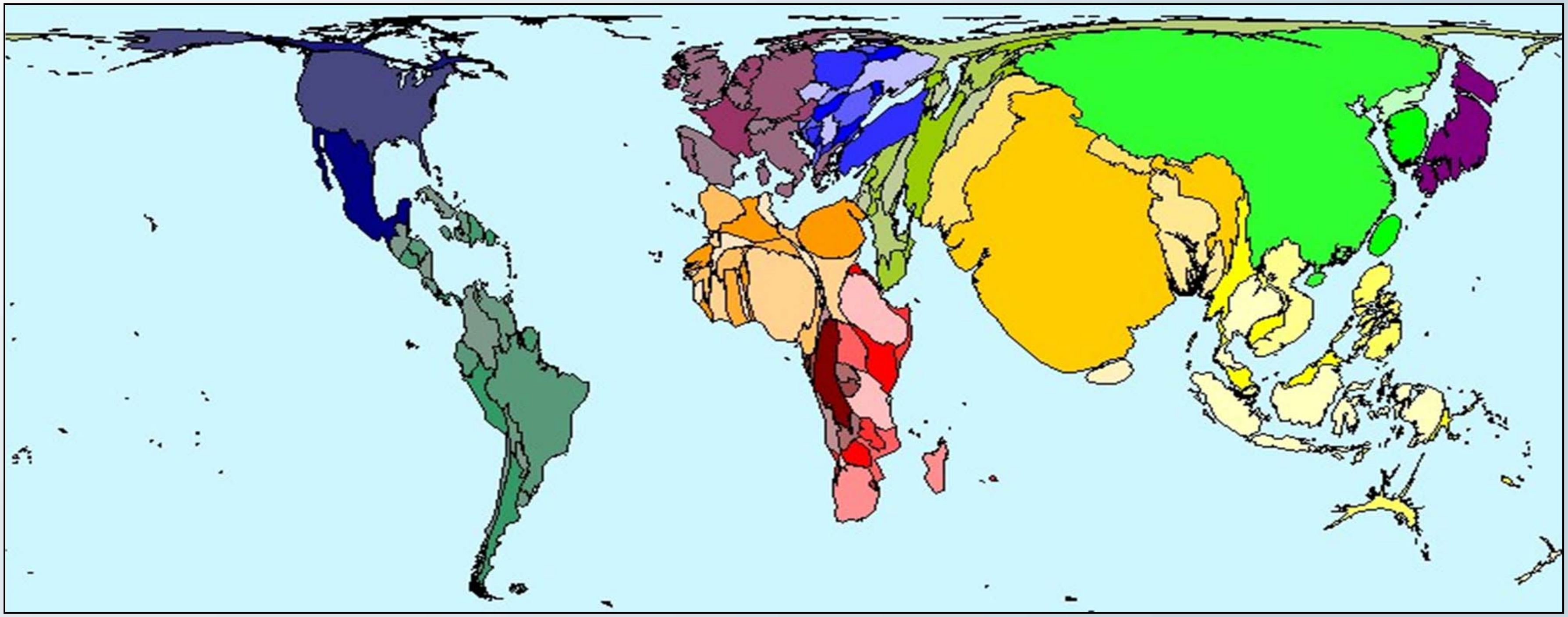
1. **Pre-industrial society**.
   Deaths high, births high.
   Stable population.
2. **Developing society**.
   Deaths decrease, births high.
   Growing population.
3. **Transition**.
   Deaths low, births decrease.
   Growing population.
4. **Developed society**.
   Deaths low, births low.
   Stable population.

Factors affecting population growth

	Births	*Deaths*	*Growth*
• Food supply	↑	↓	↑
• Hygiene, sanitation		↓	↑
• Medical care		↓	↑
• Religious beliefs	↑ *(if at all)*		↑
• Epidemics		↑	↓
• Econ development	↓	↓	↓
• Education	↓	↓	↓
• Status of women	↓		↓
• Contraceptives	↓		↓
• Gov't incentives	↑	*or* ↓	*either*
• Em- or immigration	*n/a*	*n/a*	*either*

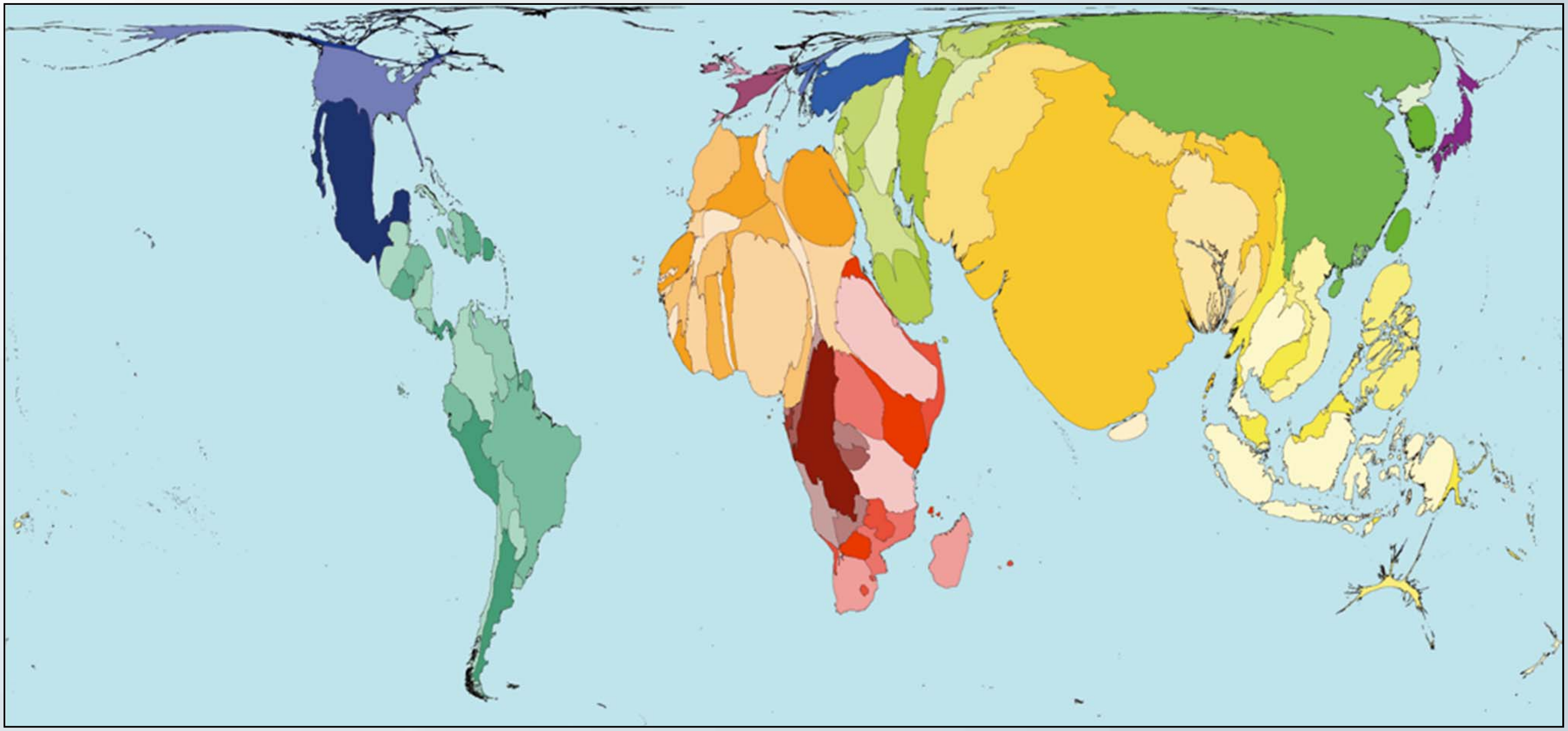
Effect on ***r***, the net growth rate of the population

Where is the world's population?



North America:	540M (8%)	Africa:	1000M (14%)
Latin America:	580M (8%)	Asia:	4100M (59%)
Europe:	740M (11%)	Oceania:	30M (0.4%)

Where is the world's population growth?



Examples of low growth rates		Examples of high growth rates	
Italy	**0.07%** per year	India	**1.58%** per year
United States	**0.88%** per year	Nigeria	**2.02%** per year
Argentina	**1.07%** per year	DR Congo	**3.22%** per year

IPAT

Quantifying human impacts: using the IPAT identity to build scenarios

- The **IPAT** method (Ehrlich & Holdren, 1972) is a simple but useful model for building scenarios of environmental impact:

  I = P • A • T

  Impact = Population × Affluence × Technology

  = (persons) × (GDP/person) × (impact/GDP)

- "Technology" refers to the technical efficiency of economic activity with regard to environmental impacts -- the more efficient the "technology," the lower the impact per $ of GDP
- Key Observation: Impact increases as the *product* – not the *sum* – of the three contributing factors P, A, T
- Example: If population, affluence, and technology all double, the impacts will be 2 × 2 × 2 = 8 times their initial magnitude!!

Sample IPAT calculation

Estimate the amount of CO_2 emitted annually by personal vehicles in the United States.

Population: 310,000,000 persons

Affluence: 11,000 vehicle-miles / person · year

Technology: 0.4 kg CO_2 / vehicle-mile

$$\left(3.1\times10^{8}\,people\right)\cdot\left(\frac{1.1\times10^{4}\,vehicle\ miles}{person\cdot year}\right)\cdot\left(\frac{0.4\,kg\ CO_2}{vehicle\ mile}\right)=$$

1.4×10^9 t CO_2 per year

Sample IPAT Calculation (cont.)

In Kenya:

This number is much **smaller.**

This number is much **smaller.**

This number is **larger.**

$$\left(3.1\times10^{8}\,people\right)\cdot\left(\frac{1.1\times10^{4}\,vehicle\ miles}{person\cdot year}\right)\cdot\left(\frac{0.4\ kg\ CO_2}{vehicle\ mile}\right)=$$

...so the overall impact is much smaller.

Can also omit <u>population</u> to calculate <u>per capita</u> impact.

Implications of IPAT

In general, three factors determine the impact of humans on their environment:

1. Population size
2. Consumption / affluence
3. Technology

Therefore, to control environmental impacts, we can:

1. Control population growth
2. Control consumption
3. Improve technologies

All these approaches are used in environmental policy.

Carbon Emissions Calculation

Impacts can be disaggregated into more than 3 drivers, e.g.

$$C = P \times GDP/P \times E/GDP \times C/E$$

- C = carbon emissions (kg)
- P = population (pers)
- GDP/P = per capita economic activity ($/pers)
- E/GDP = energy intensity of economic activity (GJ/$)
- C/E = carbon intensity of energy supply (kg/GJ)

Example In the year 2000:

- P = 6.1 billion persons
- GDP/P = $7400/pers
- E/GDP = 0.01 GJ/$
- C/E = 14 kgC/GJ

Calculate the kg and Gt of carbon emissions.

Ex. CO_2 Emissions Calculation

Calculate the kg and Gt of carbon emissions.

$$? \text{ kg C emissions} = 6.1 \times 10^9 \cancel{\text{pers}} \left(\frac{7400 \cancel{\$}}{1 \cancel{\text{pers}}}\right)\left(\frac{0.01 \cancel{\text{GJ}}}{1 \cancel{\$}}\right)\left(\frac{14 \text{ kg C}}{1 \cancel{\text{GJ}}}\right)$$

$$= \mathbf{6.3 \times 10^{12}} \cancel{\textbf{kg}} \textbf{ C} \left(\frac{1 \cancel{\text{t}}}{10^3 \cancel{\text{kg}}}\right)\left(\frac{1 \text{ Gt}}{10^9 \cancel{\text{t}}}\right) = \mathbf{6.3 \text{ Gt C}}$$

But what if these quantities are changing?

- Annual growth is not exponential growth, but annual growth can be represented approximately as "exponential growth" when the growth rate is small (< ~10%/y)
- The total rate of growth for the output can be viewed as the sum of the growth rates for population, GDP/pers, E/GDP, and C/E.

Ex. Carbon Emissions Growth Rate

- According to the World Bank the world **population (P)** is growing at about 1.2% per year and is gradually declining over time.
- **GDP per person (g)** worldwide is growing at about 1.5% per year.
- **Energy intensity of GDP (i)** was declining worldwide for decades at about -1% per year…getting 1% more efficient
- **Carbon intensity of energy use (c)** was declining about -0.3% per year
- Carbon emissions growth rate based on this model:

$$r = 1.2\% + 1.5\% - 1.0\% - 0.3\% = 1.4\%/y$$

$$C(t) = C_0 e^{rt} = C_0 e^{0.014t}$$

Ex. Carbon Emissions Growth Rate

- If the anthropogenic carbon released into the atmosphere now is 9 Gt per year, if we assume exponential growth and unchanged rates of change for population, GDP/pers, GJ/$, and kg C/ GJ, what would be the mass of carbon released into to atmosphere be in ten years?

  $$C(t) = C_0 e^{0.014t} = 9 \text{ Gt } e^{0.014\ 1/y(10\ y)} \approx 10 \text{ Gt}$$

  What's the doubling time?

  $$T_d = 0.693/0.014 \approx 50 \text{ years}$$

- What do the numbers tell us? Why should we be careful in applying these numbers?